Veer Surendra Sai University of Technology, Burla

Department of Electrical & Electronics Engg

Subject: Signals and Systems-I

Subject code: BEE-1605

Branch(semester): EEE (5^{th} sem)

SYLLABUS OF SIGNALS & SYSTEMS-I (3-1-0)

MODULE-I (10 HOURS)

Introduction of Signals, Classification of Signals, General Signal Characteristics, Signal energy & Power, Continuous-Time Signals , Discrete-Time Signals

Basic System Properties, Systems with and without memory, Invertibility, casuality, Stability, Time invariance, Linearity, Linear Time Invariant (LTI) Systems, Discrete Time LTI Systems, Convolution

Representation of Linear Time-Invariant Discrete-Time Systems Convolution of Discrete-Time

Signals Convolution Representation of Linear Time-Invariant Continuous-Time Systems Convolution of Continuous-Time Signals, Properties of LTI Systems, Casual systems

MODULE-II (10 HOURS)

Fourier Representations for Signals: Representation of Discrete Time Periodic signals, Continuous Time Periodic Signals, Discrete Time Non Periodic Signals, Continuous Time Non-Periodic Signals, Properties of Fourier Representations,

Frequency Response of LTI Systems, Fourier Transform representation for Periodic and discrete time Signals, Sampling, reconstruction, Discrete Time Processing of Continuous Time Signals, Fourier Series representation for finite duration Nonperiodic signals.

MODULE-III (10 HOURS)

Modulation Types and Benefits, Full Amplitude Modulation, Pulse Amplitude Modulation, Multiplexing, Phase and Group delays

Representation of Signals using Continuous time Complex Exponentials: Laplace Transform, Unilateral Laplace Transform, its inversion, Bilateral Laplace Transform, Transform Analysis of Systems

MODULE-IV (10 HOURS)

Representation of Signals using Discrete time Complex Exponentials: The Z-Transform, Properties of Region of convergence, Inverse Z-Transform, Transform Analysis of LTI Systems, Unilateral Z Transform.

BOOKS

[1] Simon Haykin and Barry Van Veen, *“Signals and Systems”,* John Wiley & Sons.

[2] Alan V. Oppenheim, Alan S. Willsky, with S. Hamid, S. Hamid Nawab, *“Signals and Systems”,* PHI.

[3] Hwei Hsu, *“Signals and Systems”,* Schaum's Outline TMH

[4] Edward w. Kamen and Bonnie s. Heck, *“Fundamentals of Signals & systems using Web and MATLAB”*,

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Contents

Introduction to Signals:

What is a Signal ?

- A signal is formally defined as a function of one or more variables that conveys information on the nature of a physical phenomenon.
- When the function depends on a single variable, the signal is said to be one dimensional. E.g.; Speech signal (Amplitude varies with respect to time)
- When the function depends on two or more variables, the signal is said to be multidimensional. E.g.; Image – 2D (Horizontal & vertical coordinates of the images are two dimensional)

What is a System ?

- A system is formally defined as an entity that manipulates one or more signals to accomplish a function, thereby yielding new signals.

e.g.; In a communication system the input signal could be a speech signal or computer data. The system itself is made up of the combination of a transmitter, channel and a receiver. The output signal is an estimate of the information contain in the original message.

The examples of other systems are control systems, biomedical signal processing system, audio system, remote sensing system, microelectro mechanical system etc.

General signal characteristics:

(a) *Multichannel & multidimensional signals:*

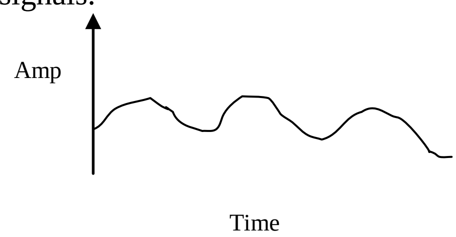
- A signal is described by a function of one or more independent variables.
- The value of the function (dependent variable) can be real valued scalar quantity, a complex valued quantity or perhaps a vector.

  Real valued signal $x_1(A) = A\,sin3\pi t$

  Complex valued signal $x_2(A) = Ae^{j3\pi t} = A\,cos3\pi t + jA sin3\pi t$
- In some applications, signals are generated by multiple sources or multiple sensors. Such signals can be represented in vector form and we refer such a vector of signal as a multichannel signal.

  E.g.; In electrocardiography, 3-lead & 12-lead electrocardiograms (ECG) are often used, which result in 3-channel & 12-channel signals.

One dimensional: If the signal is a function of a single independent variable, the signal is called 1-D signal.
e.g.; Speech signal

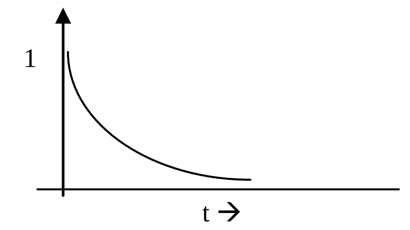


Multidimensional signal: Signals can be functions of more than one variable, e.g., image signals (2D), Colour image (3D), etc.

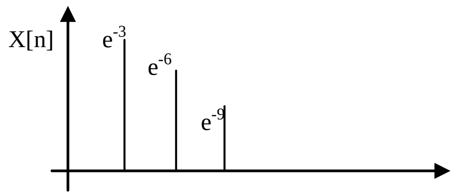
Classification of signals

Broadly we classify signals as:

1. ***Continuous-time signal***: A signal $x(t)$, is said to be continuous-time signal if it is defined for all time t, where t is a real-valued variable denoting time.
   Ex: $x(t) = e^{-3t}u(t)$



   Discrete-time signal: A signal $x(n)$, is said to be discrete-time signal; if it is defined only at discrete instant of time, where n is an integer-valued variable denoting the discrete samples of time. We use square brackets $[\cdot]$ to denote a discrete-time signal.
   Ex: $x[n] = e^{-3n}u[n]$

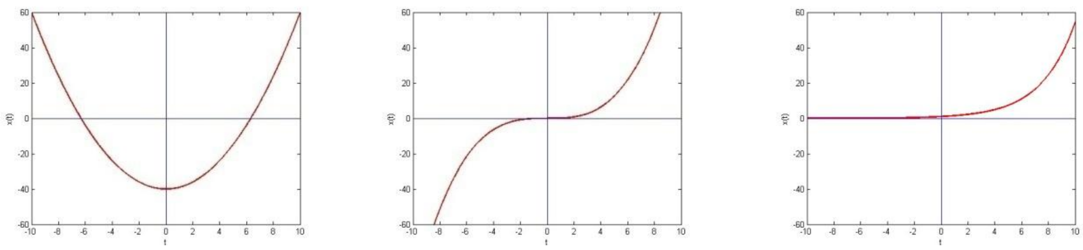


2. ***Even and odd signal***:

A continuous-time signal $x(t)$ is even, if $x(-t) = x(t)$ and it is odd if $x(-t) = -x(t)$.

A discrete-time signal $x[n]$ is even if $x[-n] = x[n]$ and is odd if $x[-n] = -x[n]$.

Example 1: $x(t) = t^2 - 40$ is even.
Example 2: $x(t) = 0.1t^3$ is odd.
Example 3: $x(t) = e^{0.4t}$ is neither even nor odd.



(a) (b) (c)

Figure: Illustrations of odd and even functions. (a) Even; (b) Odd; (c) Neither.

Decomposition Theorem

Every continuous-time signal $x(t)$ can be expressed as:

$$x(t) = y(t) + z(t)$$

where $y(t)$ is even, and $z(t)$ is odd.

$$y(t) = \frac{x(t) + x(-t)}{2}$$

and

$$z(t)\mathrm{m} = \frac{x(t) - x(-t)}{2}$$

3. *Periodic & non-periodic signals*:

A continuous time signal $x(t)$ is periodic if there is a constant $T > 0$, such that

$x(t) = x(t + T)$, for all t

A discrete time signal $x[n]$ is periodic if there is an integer constant $N > 0$, such that

$x[n] = x[n + N]$, for all n

Signals do not satisfy the periodicity conditions are called non-periodic signals.

Note: The smallest value of T (N) that satisfies the above equations is called fundamental period

Example: Determine the fundamental period of the following signals:

(a) $e^{j3\pi t/5}$

(b) $e^{j3\pi n/5}$

Solution:

(a) Let $x(t) = e^{j3\pi t/5}$. If $x(t)$ is a periodic signal, then there exists $T > 0$ such that $x(t) = x(t + T)$. Therefore,

$$x(t) = \mathrm{x}(t + T)$$

$$\Rightarrow e^{j3\pi t/5} = e^{j3\pi(t+T)/5}$$

$$\Rightarrow 1 = e^{j3\pi T/5}$$

$$\Rightarrow e^{j2k\pi} = e^{j3\pi T/5}$$

$$\Rightarrow T = \frac{10}{3} \qquad (k = 1)$$

(b) L et $\mathrm{x}[\mathrm{n}] = \mathrm{e}^{j3\pi n/5}$. If $\mathrm{x}[\mathrm{n}]$ is a periodic signal, then there exists an integer $\mathrm{N} > 0$ such that $x[n] = x[n + N]$. So,

$$x[n] = x[n + N]$$

$$\Rightarrow e^{j3\pi n/5} = e^{j3\pi(n+N)/5}$$

$$\Rightarrow 1 = e^{j3\pi N/5}$$

$$\Rightarrow e^{j2k\pi} = e^{j3\pi N/5}$$

$$\Rightarrow T = 10 \qquad (\mathrm{k} = 3)$$

4. Energy signals and power signals:

In electrical systems, a signal may represent a voltage or a current. Consider a voltage v(t)developed across a resistor R, producing a current i(t). The instantaneous power dissipated in this resistor is defined by

Define the *total energy* of the continuous-time signal *x(* t) as

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)dt$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} x^2(t)dt$$

and its *time-averaged,* or *average, power* as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$$

From above equation, we readily see that the time-averaged power of a periodic signal *x(t)* of fundamental period T is given by

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$$

The square root of the average power *P* is called the *root mean-square* (rms) value of the periodic signal *x(* t).

In the case of a discrete-time signal *x[n],* the integrals in above equations are replaced by corresponding sums. Thus, the total energy of *x[n]* is defined by

$$E = \sum_{n=-\infty}^{\infty} x^2[n]$$

and its average power is defined by

$$P = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n]$$

A signal is referred to an energy signal if and only if the total energy is finite .i.e.,

$0 < E < \infty$

A signal is referred to an power signal if and only if the average power is finite .i.e.,

$0 < P < \infty$

Note: Energy signal has zero time average power and power signal has infinite energy.

Example: $x(n) = (-0.5)^n u[n]$

Solution:

$$E = \sum_{n=-\infty}^{\infty} x^2[n] = \sum_{n=0}^{\infty} 0.25^n = \frac{1}{1-0.25} = \frac{4}{3} < \infty$$

$$P = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n] = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 0.25^n = \frac{1}{2\infty+1} \sum_{n=0}^{N} 0.25^n = 0$$

We got power zero and finite energy. Hence it is an energy signal.

5. *Deterministic signals and random signals:*

The deterministic signal is a signal about which there is no uncertainty with respect to its value at any time. The deterministic signals may be modeled as completely specified function of time.
Example: $x(t) = \cos^2(2\pi t)$
A random signal is a signal about which there is uncertainty before it occurs. Example: The electrical noise generated in the amplifier of a radio or television receiver.

Basic Operations of Signals

Operation performed on independent variable:

Time Shift

For any t_0 and n_0 , time shift is an operation defined as

$$x(t) \rightarrow x(t - t_0)$$
$$x[n] \rightarrow x[n - n_0].$$

If $t_0 > 0$, the time shift is known as “delay”. If $t_0 < 0$, the time shift is known as “advance”.

Example. In Fig. given below, the left image shows a continuous-time signal $x(t)$. A time- shifted version $x(t - 2)$ is shown in the right image.

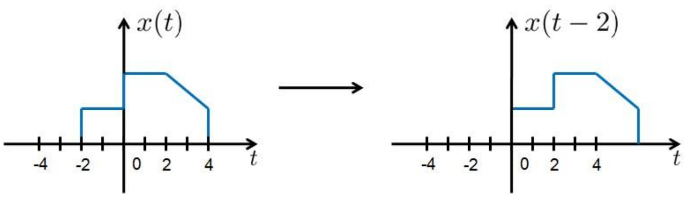


Figure: An example of time shift.

Time Reversal

Time reversal is defined as

$$x(t) \rightarrow x(-t)$$
$$x[n] \rightarrow x[-n],$$

which can be interpreted as the “flip over the y-axis”.

Example:

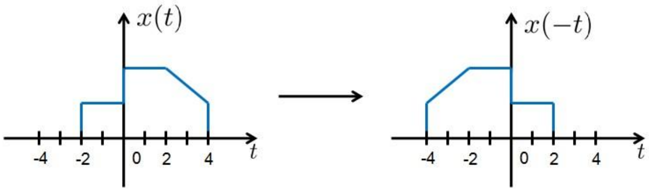


Figure: An example of time reversal.

Time Scaling

Time scaling is the operation where the time variable t is multiplied by a constant a:

$$x(t) \rightarrow x(at), \quad a > 0$$

If a > 1, the time scale of the resultant signal is "*decimated*" (speed up).

If 0 < a < 1,
the time scale of the resultant signal is "*expanded*" (slowed down).

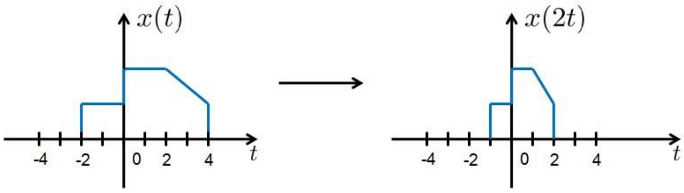


Figure : An example of time scaling.

Decimation and Expansion

Decimation and expansion are standard discrete-time signal processing operations.

Decimation is defined as

$$y_D[n] = x[Mn], \text{ for some integers } M$$

Where, M is the decimation factor.

Expansion is defined as

$$y_E[n] = \begin{cases} x[\frac{n}{L}], & n = \text{integer multiple of } L \\ 0, & \text{otherwise.} \end{cases}$$

Where, L is the expansion factor.

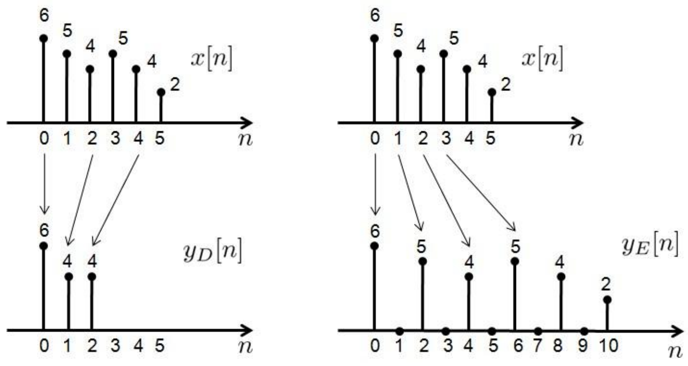


Figure 1.8: Examples of decimation and expansion for M = 2 and L = 2.

Combination of Operations

Generally, linear operation (in time) on a signal x(t) can be expressed as $y(t) = x(at-b)$. The recommended method is "Shift, then Scale".

Example: The signal x(t) shown in Figure of sketch $x(3t-5)$.

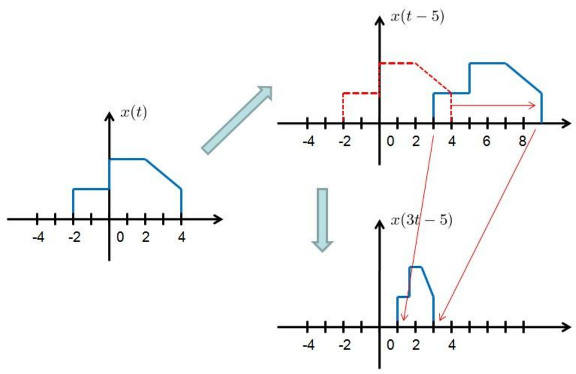


Figure: An example of Shift, then Scale

Operation performed on dependent variable:

Amplitude scaling:

Let x(t) denote a continuous time signal

By amplitude scaling, we get y(t) = cx(t)

Where, c is the scaling factor.

Example: An electronic amplifier, a device that performs amplitude scaling.

For discrete time signal $y[n] = cx[n]$

Amplitude addition:

Let $x_1(t)$ and $x_2(t)$ is a pair of continuous time signal

By adding these two signals, we get $y(t) = x_1(t) + x_2(t)$

Example: An audio mixture

For discrete time signal, $y[n] = x_1[n] + x_2[n]$

Amplitude multiplication:

Let $x_1(t)$ and $x_2(t)$ is a pair of continuous time signal

By multiplying these two signals, we get $y(t) = x_1(t)\, x_2(t)$

Example: An AM radio signal, in which

$x_1(t)$ is an audio signal

$x_2(t)$ is an sinusoidal carrier wave

For discrete time signal, $y[n] = x_1[n]\, x_2[n]$

Differentiation:

$y(t)\frac{d}{dt}\, x(t)$

Example: Voltage across an inductor L → $v(t)\frac{d}{dt}\, i(t)$

Integration:

$y(t)=\int_{-\infty}^{t} x(\tau)d\tau$

Example: Voltage across a capacitor C → $y(t)=\frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$

Elementary Signals

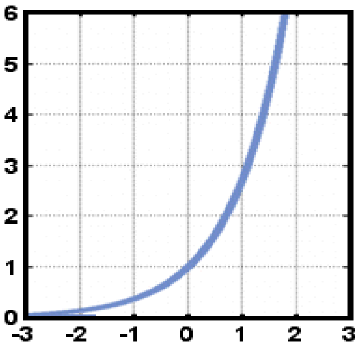
Several elementary signals feature prominently in the study of signals and systems. These are exponential and sinusoidal signals, the step function, the impulse function, and the ramp function, all of which serve as building blocks for the construction of more complex signals

Exponential Signals

A real exponential signal, in its most general form, is written as

$x(t) = Be^{at}$,

where both B and a are real parameters. The parameter B is the amplitude of the exponential signal measured at time $t = 0$. Depending on whether the other parameter a is positive or negative, we may identify two special cases:



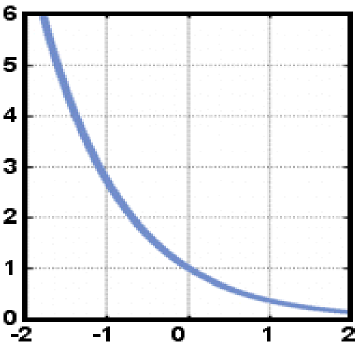
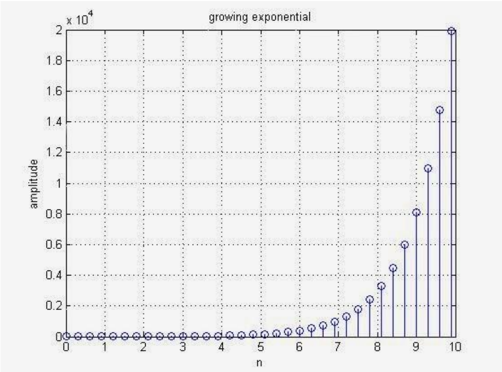


Fig: Growing exponential, for $a > 0$ *Decaying exponential,* for $a < 0$

In discrete time, it is common practice to write a real exponential signal as $x[n] = Br^n$



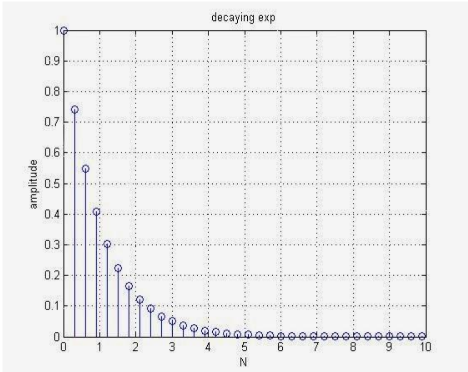


Fig: Growing exponential for r > 1 *Decaying exponential for 0 < r <1*

Impulse functions

The discrete-time version of the *unit impulse* is defined by

$$\delta\,[n] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$$

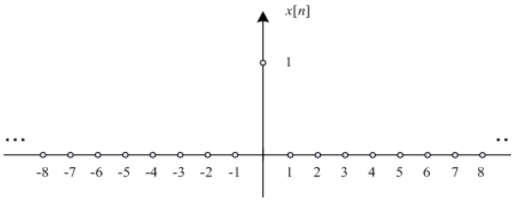


Fig: Discrete time form of unit impulse

The continuous-time version of the unit impulse is defined by the following pair of relations:

$$\delta\,(t) = 0 \text{ for } t \neq 0$$

and $\int_{-\infty}^{\infty} \delta\,(t)dt = 1$

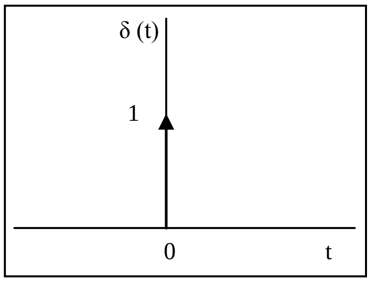


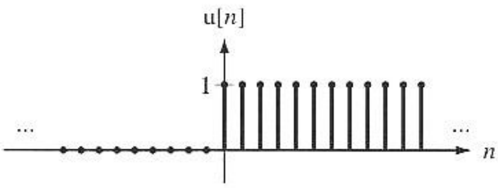
Fig: Continuous time form of unit impulse

Above equation says that the impulse δ *(t)* is zero everywhere except at the origin. Equation says that the total area under the unit impulse is unity. The impulse δ *(t)* is also referred to as the *Dirac delta function.*

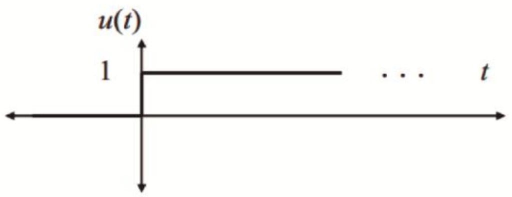
Step function:

The discrete-time version of the unit- step function is defined by:

$$\boxed{u[n] = \begin{cases} 1\,, n \geq 0 \\ 0\,, n < 0 \end{cases}}$$



The continuous-time version of the unit- step function is defined by: $u(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$



Ramp function:

The integral of the step function u(t) is a ramp function of unit slope.

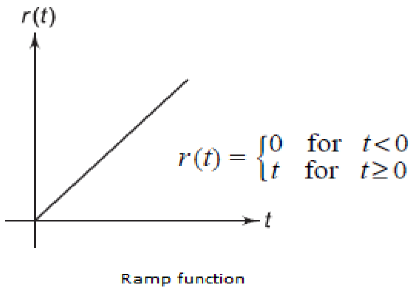
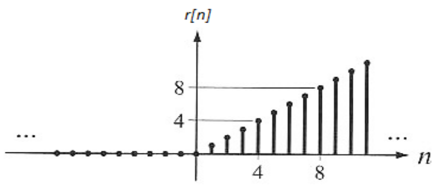


Fig: Ramp function of unit slope

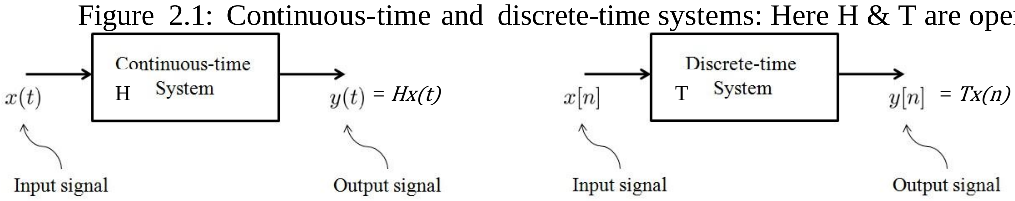
The discrete-time version of the unit- ramp function is defined by: $r[n] = \begin{cases} n, & n\geq 0 \\ 0, & n<0 \end{cases}$



Introduction to Systems

- Systems are used to process signals to allow modification or extraction of additional information from the signal.
- A system may consist of physical components (hardware realization) or an algorithm (operator) that computes the output signal from the input signal.
- A physical system consists of inter-connected components which are characterized by their input-output relationships.

Figure 2.1: Continuous-time and discrete-time systems: Here H & T are operators.



Properties of systems: (Classification of systems):

1 Static (Memoryless) & Dynamic (with memory):

Static: A system is static if the output at time t (or n) depends only on the input at time t (or n).

Examples:

1. $y(t) = (2x(t) - x^2(t))^2$ is memoryless, because y(t) depends on x(t) only. There is no $x(t-1)$, or $x(t+1)$ terms, for example.
2. $y[n] = x^2[n]$ is memoryless. In fact, this system is passing the input to output directly, without any processing.
3. Current flowing through a resistor i.e., $i(t) = \frac{1}{R} v(t)$

Dynamic: A system is said to possess memory if its output signal depends on past or future values of input.

Example:

1. Inductor and capacitor, since the current flowing through the inductor at time 't' depends on the all past values of the voltage v(t) i.e., $i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$ and $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$
2. The moving average system given by $y(n)= \frac{1}{3}(x[n]+x[n-1]+x[n-2])$

2 Stable & unstable system:

- A system is said to be bounded-input, bounded-output (BIBO) stable if and only if every bounded input results in a bounded output, otherwise it is said to be unstable.
- If for $|x(t)| \le M_x < \infty$ for all t, output is $|y(t)| \le M_y < \infty$ for all t; where M_x & M_y are some finite positive number.

Example:
1. $y(t) = x(t-3)$ is a stable system.
2. $y(t) = t\,x(t)$ is an unstable system.
3. $y[n] = e^{x[n]}$ is a stable system.

Assume that $|x(n)| \le M_x < \infty$, for all 't'

$$y[n] = e^{x[n]} = e^{Mx} = \text{finite} \rightarrow \text{Stable}$$

4. $y[n] = r^n x[n]$, where $r > 1$

Assume that $|x(n)| \le M_x < \infty$, for all 't', then

$$|y[n]| = |r^n x[n]| = |r^n|\,|x[n]|$$

as 'n' $\rightarrow \infty$ $|r^n| \rightarrow \infty$

so $y[n] \rightarrow \infty$ hence unstable.

3 Causal and non-Causal system:

Causal: A system is said to be causal if the present value of output signal depends only on the present or past values of the input signal. A causal system is also known as physical or non-anticipative system.

Example:1. The moving average system given by $y(n)= \frac{1}{3}(x[n]+x[n-1]+x[n-2])$

2. $y(t) = x(t)\cos 6t$

Note: i) Any practical system that operates in real time must necessarily be causal.
ii) All static systems are causal.

Non-Causal: A system is said to be non-causal if the present value of output signal depends on one or more future values of the input signal.

Example:1. The moving average system given by $y(n)=\frac{1}{3}(x[n]+x[n-1]+x[n+2])$

4 Time invariant and time variant system:

Time invariant: A system is time-invariant if a time-shift of the input signal results in the same time-shift of the output signal.

That is, if

$$x(t) \to y(t),$$

then the system is time-invariant if

$$x(t-t_0) \to y(t-t_0), \text{ for any } t_0.$$

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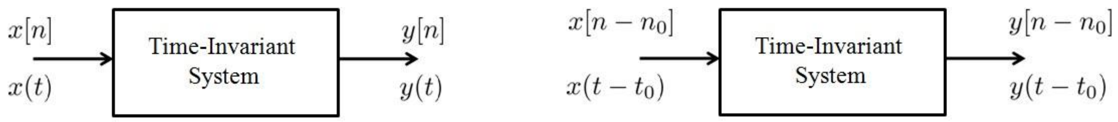


Figure 2.2: Illustration of a time-invariant system.

Example 1.
The system $y(t) = \sin[x(t)]$ is time-invariant

Proof. Let us consider a time-shifted signal $x_1(t) = x(t-t_0)$. Correspondingly, we let $y_1(t)$ be the output of $x_1(t)$. Therefore,

$$y_1(t) = \sin[x_1(t)] = \sin[x(t-t_0)].$$

Now, we have to check whether $y_1(t) = y(t-t_0)$. To show this, we note that

$$y(t-t_0) = \sin[x(t-t_0)],$$

which is the same as $y_1(t)$. Therefore, the system is time-invariant. □

Time variant: A system is time-variant if its input –output characteristic changes with time.

Example 2:
The system $y[n] = nx[n]$ is time-variant.

Proof: Output for a time shifted input is

$$y[n]\,|_{\,x(n-k)} = nx(n-k)$$

then the same time shifted output is

$$y(n-k) = (n-k)x(n-k)$$

the above two equations are not same. Hence it is time variant.

4 Linear and non-linear system:

Linear system: A system is said to be linear if it satisfies two properties i.e.; superposition & homogeneity.

Superposition: It states that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of responses (Outputs of the system to each of the individual input signal.

For an input $x(t) = x_1(t)$, the output $y(t) = y_1(t)$

and input $x(t) = x_2(t)$, the output $y(t) = y_2(t)$

then, the system is linear if & only if

$$\mathcal{T}[a_1x_1(t) + a_2x_2(t)] = a_1\mathcal{T}[x_1(t)] + a_2\mathcal{T}[x_2(t)]$$

Homogeneity: If the input x(t) is scaled by a constant factor 'a', then the output y(t) is also scaled by exactly the same constant factor 'a'.

For an input $x(t) \rightarrow$ output $y(t)$

and input $x_1(t) = ax(t) \rightarrow$ output $y_1(t) = ay(t)$

Example 1:

The system $y(t) = 2\pi x(t)$ is linear. To see this, let's consider a signal

$$x(t) = ax_1(t) + bx_2(t),$$

where $y_1(t) = 2\pi x_1(t)$ and $y_2(t) = 2\pi x_2(t)$. Then

$$\begin{aligned} ay_1(t) + by_2(t) &= a(2\pi x_1(t)) + b(2\pi x_2(t)) \\ &= 2\pi[ax_1(t) + bx_2(t)] = 2\pi x(t) = y(t). \end{aligned}$$

Example 2.

The system $y[n] = (x[2n])^2$ is not linear. To see this, let's consider the signal

$$x[n] = ax_1[n] + bx_2[n],$$

where $y_1[n] = (x_1[2n])^2$ and $y_2[n] = (x_2[2n])^2$. We want to see whether $y[n] = ay_1[n] + by_2[n]$. It holds that

$$ay_1[n] + by_2[n] = a(x_1[2n])^2 + b(x_2[2n])^2.$$

However,

$$y[n] = (x[2n])^2 = (ax_1[2n] + bx_2[2n])^2 = a^2(x_1[2n])^2 + b^2(x_2[2n])^2 + 2abx_1[n]x_2[n].$$

5 Invertible and non-invertible system:

A system is said to be invertible if the input of the system can be recovered from the output.

Let the set of operations needed to recover the input represents the second system which is connected in cascade with the given system such that the output signal of the second system is equal to the input applied to the given system.

Let H → the continuous time system

$x(t)$ → input signal to the system

$y(t)$ → output signal of the system

H^{inv} → the second continuous time system

$$x(t) \longrightarrow \boxed{H} \xrightarrow{y(t)} \boxed{H^{inv}} \xrightarrow{x(t)}$$

The output signal of the second system is given by

$$H^{inv}\{y(t)\} = H^{inv}\{Hx(t)\}$$
$$= H^{inv}H\{x(t)\}$$

For the output signal to equal to the original input, we require that

$$\boxed{H\,H^{inv} = I}$$

Where 'I' denotes the identity operator.

The system whose output is equal to the input is an identity system. The operator H^{inv} must satisfy the above condition for H to be an invertible system. Cascading a system, with its inverse system, result in an identity system.

Example:

An inductor is described by the relation

$$y(t) = \frac{1}{L}\int_{-\infty}^{t} x(\tau)d\tau \text{ is an invertible system}$$

because, by rearranging terms, we get

$$x(t) = L\frac{d}{dt}\,y(t),$$

which is the inversion formula.

Note:i) A system is not invertible unless distinct inputs applied to the system produce distinct outputs.

ii) There must be a one to one mapping between input and output signal for system to be invertible.

Non-invertible System: When several dibfferent inputs results in the same output, it is impossible to obtain the input from output. Such system is called a non-invertible system.

Example: A square-law system described by the input output relation

$$y(t) = x^2(t), \text{ is non-invertible,}$$

because distinct inputs x(t) & -x(t) produce the same output y(t) [not distinct output].

Linear –time convolution system (LTI)

Linear time invariant (LTI) systems are good models for many real-life systems, and they have properties that lead to a very powerful and effective theory for analyzing their behavior. The LTI systems can be studied through its characteristic function, called the impulse response. Further, any arbitrary input signal can be decomposed and represented as a weighted sum of unit sample sequences. As a consequence of the linearity and time invariance properties of the system, the response of the system to any arbitrary input signal can be expressed in terms of the unit sample response of the system. The general form of the expression that relates the unit sample response of the system and the arbitrary input signal to the output signal, called the convolution sum, is also derived.

Resolution of a Discrete-time signal into impulses:

Any arbitrary sequence x(n) can be represented in terms of delayed and scaled impulse sequence δ(n). Let x(n) is an infinite sequence as shown in figure below.

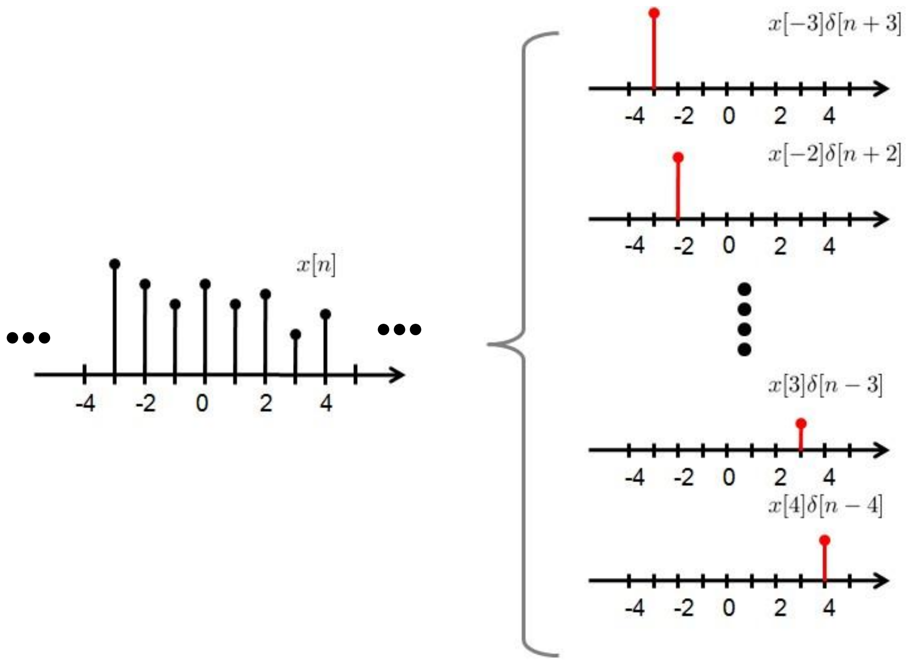


Figure 1.13: Representing of a signal *x[n]* using a train of impulses δ[n − k].

The sample x(0) can be obtained by multiplying x(0), the magnitude, with unit impulse δ(n)

i.e., $x[n]\,\delta[n] = \begin{cases} x(0), & n=0 \\ 0, & n\neq 0 \end{cases}$

Similarly, the sample x(-3) can be obtained as shown in the figure.

i.e., $x[-3]\,\delta[n+3] = \begin{cases} x(-3), & n=-3 \\ 0, & n\neq -3 \end{cases}$

In the same way we can get the sequence *x[n]* by summing all the shifted and scaled impulse function

i.e., $x[n] = \ldots\; x[-3]\,\delta[n+3] + x[-2]\,\delta[n+2] + \ldots + x[0]\,\delta[n] + \ldots + x[4]\,\delta[n-4] \ldots$

$= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$

Impulse response and convolution sum:

Impulse response: A discrete-time system performs an operation on an input signal based on predefined criteria to produce a modified output signal. The input signal x[n] is the system excitation, and y[n] is the system response. The transform operation is shown in the figure below.

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]=T[x[n]]$$

If the input to the system is the unit impulse i.e., x[n] = δ[n], then the output of the system is known as impulse response represented by h[n] where

$$h[n] = T\,[\delta[n]]$$

Response of LTI system to arbitrary inputs: The convolution sum

From the above discussion, we get the response of an LTI system to an unit impulse as the impulse response h[n] i.e.,

$\delta[n] \longrightarrow h[n]$

$\delta[n-k] \longrightarrow h[n-k]$, by time invariant property

$x(k)\delta[n-k] \longrightarrow x(k)h[n-k]$, by homogeneity principle

$\sum_{k=-\infty}^{\infty} x(k)\delta[n-k] \longrightarrow \sum_{k=-\infty}^{\infty} x(k)h[n-k]$, by super position

As we know the arbitrary input signal is a weighted sum of impulse, the LHS = x[n] having a response in RHS = y[n] known as convolution summation.

i.e., $x[n] \longrightarrow y[n]$

In other words, given a signal x[n] and the impulse response of an LTI system h[n], the convolution between x[n] and h[n] is defined as

$$y[n] = \sum_{k=-\infty}^{\infty} x(k)h[n-k]$$

We denote convolution as $y[n] = x[n] * h[n]$.

- Equivalent form: Letting $m = n - k$, we can show that

$$\sum_{k=-\infty}^{\infty} x(k)h[n-k] = \sum_{m=-\infty}^{\infty} x(n-m)h[m] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Properties of convolution:

The following "standard" properties can be proved easily:

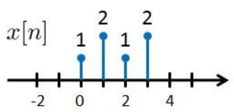
1. Commutative: $x[n] * h[n] = h[n] * x[n]$
2. Associative: $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$
3. Distributive: $x[n] * (h_1[n] + h_2[n]) = (x(t) * h_1[n]) + (x[n] * h_2[n])$

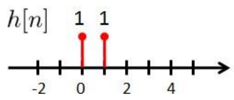
How to Evaluate Convolution?

To evaluate convolution, there are four basic steps:

1. Fold
2. Shift
3. Multiply
4. Summation

Example1: Consider the signal x[n] and the impulse response h[n] shown below.





Let's compute the output y[n] one by one. First, consider y[0]:

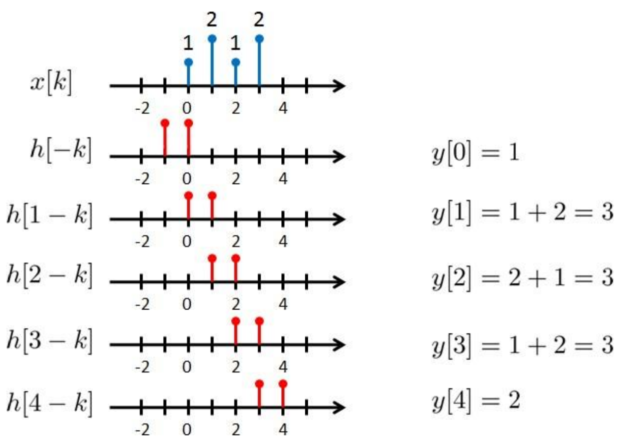
$$y[o] = \sum_{k=-\infty}^{\infty} x(k)h[0-k] = \sum_{k=-\infty}^{\infty} x(k)h[-k] = 1$$

Note that h[−k] is the flipped version of h[k], and $\sum_{k=-\infty}^{\infty} x(k)h[-k] = 1$ is the multiply-add between x[k] and h[−k].

To calculate y[l], we flip h[k] to get h[−k], shift h[−k] go get h[l−k], and multiply-add to get $\sum_{k=-\infty}^{\infty} x(k)h[1-k]$. Therefore

$$y[1] = \sum_{k=-\infty}^{\infty} x(k)h[1-k] = \sum_{k=-\infty}^{\infty} x(k)h[1-k] = 1 \times 1 + 2 \times 1 = 3$$

The calculation is shown in the figure below.



System Properties

With the notion of convolution, we can now proceed to discuss the system properties in terms of impulse responses.

Memoryless

A system is memoryless if the output depends on the current input only. An equivalent statement using the impulse response h[n] is that:

An LTI system is memoryless if and only if

$$h[n] = a\delta[n], \text{ for some } a.$$

Invertible

An LTI system is invertible if and only if there exist g[n] such that

$$h[n] * g[n] = \delta[n].$$

Causal

An LTI system is causal if and only if

$$h[n] = 0, \quad \text{for all } n < 0.$$

Stable

An LTI system is stable if and only if

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Proof: Suppose that $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$. For any bounded signal $|x[n]| \leq B$, the output is

$$|y[n]| \leq \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right|$$

$$= \sum_{k=-\infty}^{\infty} |x[k]|.|h[n-k]|$$

$$\leq B. \sum_{k=-\infty}^{\infty} |h[n-k]|$$

Therefore, $y[n]$ is bounded.

Continuous-time Convolution

The continuous-time case is analogous to the discrete-time case. In continuous-time signals, the signal decomposition is

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)d\tau$$

and consequently, the continuous time convolution is defined as

$$(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)d\tau$$

Example: The continuous-time convolution also follows the three step rule: flip, shift, multiply- add. Let us consider the signal $x(t) = e^{-at}u(t)$ for $a > 0$, and impulse response $h(t) = u(t)$. The output $y(t)$ is

Case A: $t > 0$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{-ar}\,u(\tau)\,u(t-\tau)$$

$$= \int_{0}^{t} e^{-ar}\,d\tau$$

$$= \frac{1}{-a}\,[1-e^{-at}]$$

Case B: $t \leq 0$:

$$y(t) = 0.$$

Therefore,

$$y(t) = \frac{1}{a}\,[1-e^{-at}]u(t)$$

Properties of CT Convolution

The following properties can be proved easily:

1. Commutative: $x(t) * h(t) = h(t) * x(t)$
2. Associative: $x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$
3. Distributive: $x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$

Continuous-time System Properties

The following results are analogous to the discrete-time case.

Memoryless.

An LTI system is memoryless if and only if

$$h(t) = a\delta(t), \text{ for some } a$$

Invertible.

An LTI system is invertible if and only if there exist g(t) such that

$$h(t) * g(t) = \delta(t).$$

Causal.

A system is causal if and only if

$$h(t) = 0, \quad \text{for all } t < 0$$

Stable.

A system is stable if and only if

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Interconnection of LTI systems:

1. **Parallel connection of LTI System:** Consider two LTI systems with impulse responses $h_1(t)$and $h_2(t)$ connected in parallel as shown in the figure below. The output of this connections of systems, $y(t)$, is the sum of the outputs of the two systems i.e.,

$$y(t) = y_1(t) + y_2(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

$$= x(t) * [h_1(t) + h_2(t)]$$

Identical results hold for the discrete time case.

$$x(n) * h_1(n) + x(n) * h_2(n) = x(n) * [h_1(n) + h_2(n)]$$

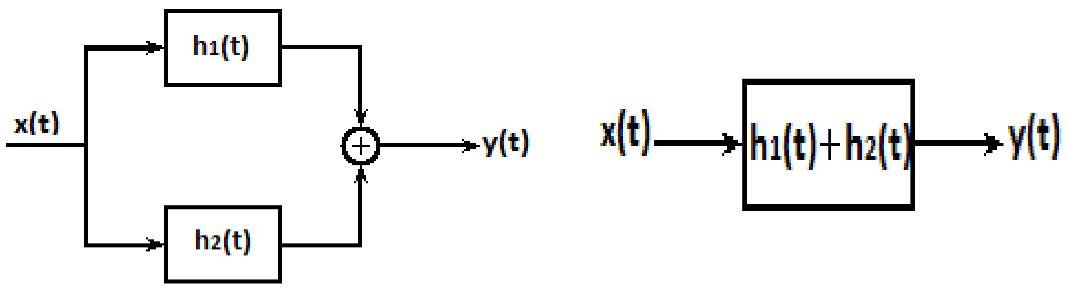


Fig: Parallel interconnection of two LTI system & its equivalent systems

2. **Cascade connection of LTI System:** Consider the cascade connection of two LTI systems as shown in the figure. The output of this connection of systems

$$y(t) = \{x(t) * h_1(t) * h_2(t)\}$$

Using associative property of convolution, we get

$$y(t) = x(t) * \{h_1(t) * h_2(t)\}$$

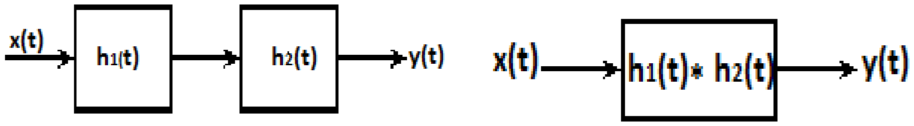


Fig: Cascade Interconnection of two LTI system & its equivalent systems

Step response: Step input response are often used to characterize the response of an LTI system to sudden changes in the input. It is defined as the output due to a unit step input signal.

Let $h[n]$ be the impulse response of a discrete-time LTI system and $s[n]$ be the step response.

Then,
$$s[n] = h[n] * u[n]$$
$$= \sum_{k=-\infty}^{\infty} h[k]u[n-k]$$

Now, as u[n-k] = 0 for k > n and u[n-k] = 1 for k ≤ n,

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]$$

i.e., the step response is the running sum of the impulse response. Similarly, the step response s(t) of a continuous-time system is expressed as the running integral of the impulse response:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

Note: These relationships may be inverted to express the impulse response in terms of the step response as

$$h[n] = s[n] - s[n-1]$$

and, $$h(t) = \frac{d}{dt} s(t)$$

Fourier Representations for Signals

In this chapter, the signal is represented as a weighted superposition of complex sinusoids. If such a signal is applied to an LTI system, then the system output is a weighted superposition of the system response to each complex sinusoid. Representing signal as superposition of complex sinusoids not only leads to a useful expression for the system output, but also provides an insightful characterization of the signals and systems. The study of signals and systems using sinusoidal representation is known as *Fourier analysis* named after Joseph Fourier.

Basing on the periodicity properties of the signal and whether the signal is discrete or continuous in time, there are four different types of Fourier representations, each applicable to a different class of signals.

Complex sinusoids and frequency response of LTI systems:

The response of an LTI system to a sinusoidal input leads to a characterization of system behavior termed as *frequency response* of the system. This characterization is obtained in terms of the impulse response by using convolution and a complex sinusoidal input signal.

Let us consider the output of a discrete-time LTI system with impulse response $h[n]$ and unit amplitude complex sinusoidal input $x[n] = e^{j\Omega n}$. This output is given by:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\Omega(n-k)}$$

We factor $e^{j\Omega n}$ from the sum to get

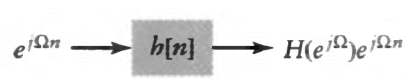
$$y[n] = e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\Omega k}$$

$$= \mathrm{H}(e^{j\Omega})e^{j\Omega n}$$

Where we have defined

$$H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\Omega k}$$

Hence, the output of the system is a complex sinusoid of the same frequency as the input, multiplied by the complex number $H(e^{j\Omega})$. The relationship is shown in figure below:



The complex scaling factor $H(e^{j\Omega})$ is not a function of time *n*, but only is a function of frequency Ω and is termed the *frequency response* of the discrete-time system.

The results obtained for continuous-time LTI system is similar to the above.

Let the impulse response of such a system be h(t) and the input be x(t) = $e^{j\omega t}$. Then the convolution integral gives the output as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}\,d\tau$$
$$= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau$$
$$= H(j\omega)e^{j\omega t},$$

Where we define,

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau.$$

The above equation is referred to as *frequency response* of the continuous time system.
Writing the complex valued frequency response $H(j\omega)$ in polar form

$$H(j\omega) = |H(j\omega)|e^{j\varphi}$$

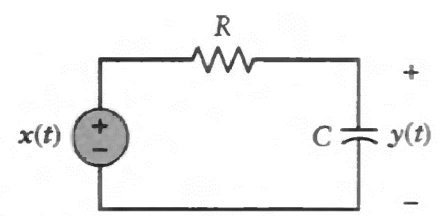
Where,

$|H(j\omega)|$ => magnitude response

And, φ => phase response = $\arg\{H(j\omega)\}$

Example: The impulse response of the system given the figure below is

$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$



Find an expression for the frequency response and plot the magnitude and phase response.

Solution: Substituting h(t) in equation of $H(j\omega)$, we get

$$
\begin{aligned}
H(j\omega) &= \frac{1}{RC}\int_{-\infty}^{\infty} e^{-\frac{\tau}{RC}}u(\tau)e^{-j\omega\tau}\,d\tau \\
&= \frac{1}{RC}\int_{0}^{\infty} e^{-\left(j\omega+\frac{1}{RC}\right)\tau}\,d\tau \\
&= \frac{1}{RC}\frac{-1}{\left(j\omega+\frac{1}{RC}\right)}e^{-\left(j\omega+\frac{1}{RC}\right)\tau}\Big|_{0}^{\infty} \\
&= \frac{1}{RC}\frac{-1}{\left(j\omega+\frac{1}{RC}\right)}(0-1) \\
&= \frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}.
\end{aligned}
$$

The magnitude response is:

$$|H(j\omega)| = \frac{\frac{1}{RC}}{\sqrt{\omega^2+\left(\frac{1}{RC}\right)^2}}$$

While the phase response is

$$\arg\{H(j\omega)\} = -\arctan(\omega RC)$$

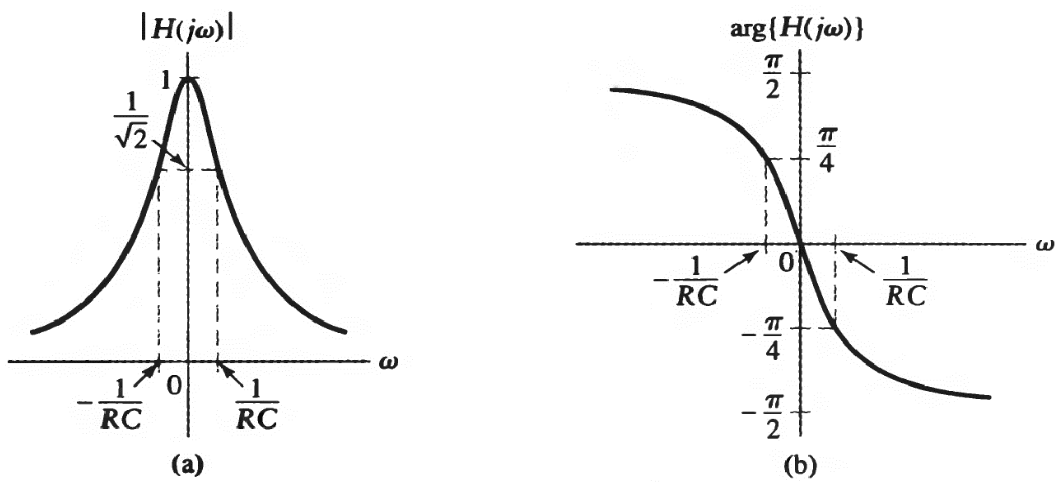


Fig: (a) Magnitude-response (b) Phase-response

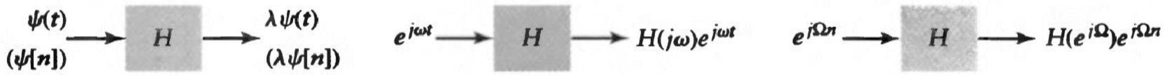
Eigenvalue and Eigenfunctions of an LTI System

Definition: For an LTI system, if the output is a scaled version of its input, then the input function is called an *eigenfunction* of the system. The scaling factor is called the *eigenvalue* of the system.

We take the complex sinusoid $\psi(t) = e^{j\omega t}$ is an *eigenfunction* of the LTI system H associated with the *eigenvalue* $\lambda = H(j\omega)$, because ψ satisfies an eigenvalue problem described by

$$H\{\psi(t)\} = \lambda\psi(t)$$

The effect of the system on an eigenfunction input signal is scalar multiplication. The output is given by the product of the input and a complex number. This eigen representation is shown in the figure below.



Fourier representation of four classes of signals:

- There are four distinct Fourier representations, each applicable to a different class of signals.
- The Fourier series (*FS*) applies to continuous time periodic signal, and the discrete - time Fourier series (*DTFS*) applies to discrete time periodic signal.
- The Fourier transform (FT) applies to a signal that is continuous in time and nonperiodic.
- The discrete-time Fourier transform (DTFT) applies to a signal that is discrete in time and nonperiodic.

Relationship between time properties of a signal and the appropriate Fourier representation is given below:

Time Property	*Periodic*	*Nonperiodic*
Continuous (t)	**Fourier Series (FS)**	**Fourier Transform (FT)**
Discrete $[n]$	**Discrete-Time Fourier Series (DTFS)**	**Discrete-Time Fourier Transform (DTFT)**

Continuous-time periodic signals: The Fourier series

Continuous-time periodic signals are represented by the Fourier series (FS). We may write the FS of a signal x(t) with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$, as

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t},$$

Where,

$$X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt$$

are the FS coefficient of the signal x(t). We say that $x(t)$ and X[k] are an FS pair and denote this relationship as

$$x(t) \xleftrightarrow{FS;\omega_o} X[k].$$

The Fourier series coefficients are known as the *frequency-domain representation of x(t)*,because each FS coefficient is associated with a complex sinusoid of different frequency.

In the representation of the periodic signal $x(t)$ by the Fourier series, the issue arises, is whether or not the series converges to $x(t)$ for each value of t, i.e., whether the signal $x(t)$ and its FS representation are equal at each value of t.

The Dirichlet conditions guarantee that the FS will be equal to $x(t)$, except at the value of t for which $x(t)$ is discontinuous. At these values of t, FS converges to the mid-point of the discontinuity. The Dirichlet conditions are:

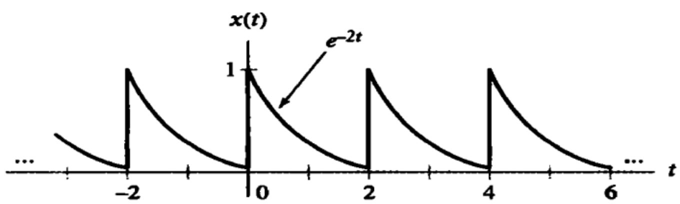
1. The signal $x(t)$ has a finite number of discontinuities in any period.
2. The signal contains a finite number of maxima and minima during any period.
3. The signal $x(t)$ is absolutely integrable (bounded) i.e.,

$$\int_T^{\infty} |x(t)|dt < \infty$$

If $x(t)$ is periodic and satisfies the Dirichlet condition, it can be represented in FS.

Direct calculation of FS coefficients:

Example: Determine the FS coefficient for signal $x(t)$



Solution: Time period T = 2, Hence, $\omega_0 = 2\pi/2 = \pi$. On the interval $0 \le t \le 2$, one period of $x(t)$ is expressed as $x(t) = e^{-2t}$. So,

$$\begin{aligned} X[k] &= \frac{1}{2}\int_0^2 e^{-2t}e^{-jk\pi t}\,dt \\ &= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\,dt. \end{aligned}$$

We evaluate the integral to get

$$\begin{aligned} X[k] &= \frac{-1}{2(2+jk\pi)}e^{-(2+jk\pi)t}\Big|_0^2 \\ &= \frac{1}{4+jk2\pi}(1-e^{-4}e^{-jk2\pi}) \\ &= \frac{1-e^{-4}}{4+jk2\pi}, \end{aligned}$$

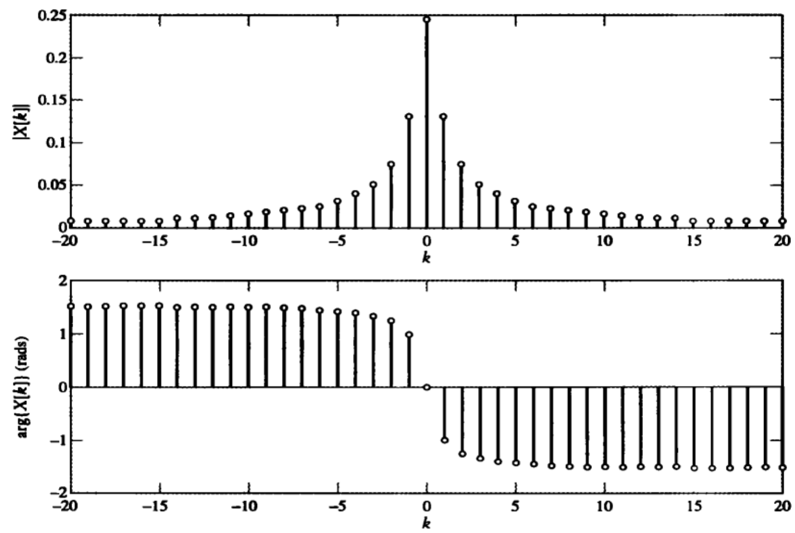


Fig: Magnitude and phase response of X[*k*]

Calculation of FS coefficients by inspection:

Example: Determine the FS representation of the signal

$$x(t) = 3\cos(\pi t/2 + \pi/4)$$

Solution: Time period $T = 4$, So, $\omega_0 = 2\pi/4 = \pi/2$. We may write FS of a signal x(t) is,

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\pi t/2}$$

Using Euler's formula to expand the cosine, gives

$$\begin{aligned} x(t) &= 3\frac{e^{j(\pi t/2+\pi/4)} + e^{-j(\pi t/2+\pi/4)}}{2} \\ &= \frac{3}{2}e^{j\pi/4}e^{j\pi t/2} + \frac{3}{2}e^{-j\pi/4}e^{-j\pi t/2} \end{aligned}$$

Equating each term in this expansion to the terms in equation of x(t) gives the FS coefficient:

$$X[k] = \begin{cases} \frac{3}{2}e^{-j\pi/4}, & k = -1 \\ \frac{3}{2}e^{j\pi/4}, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

The magnitude and phase spectra are shown below.

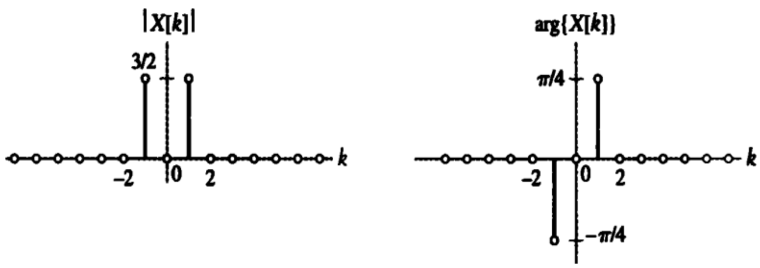


Figure: Magnitude and phase spectrum

Example: Find the time domain signal x(t) corresponding to the FS coefficient

$$X[k] = (1/2)^{|k|} e^{jk\pi/20}$$

Assume that fundamental period T = 2.

Solution: Substituting the values given for X[k] and $\omega_0 = 2\pi/2 = \pi$ into equation x(t) gives

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} (1/2)^k e^{jk\pi/20} e^{jk\pi t} + \sum_{k=-1}^{-\infty} (1/2)^{-k} e^{jk\pi/20} e^{jk\pi t} \\ &= \sum_{k=0}^{\infty} (1/2)^k e^{jk\pi/20} e^{jk\pi t} + \sum_{l=1}^{\infty} (1/2)^l e^{-jl\pi/20} e^{-jl\pi t}. \end{aligned}$$

The second geometric series is evaluated by summing from $l = 0$ to $l = \infty$, and subtracting the $l = 0$ term. The result of summing both infinite geometric series is

$$x(t) = \frac{1}{1 - (1/2)e^{j(\pi t + \pi/20)}} + \frac{1}{1 - (1/2)e^{-j(\pi t + \pi/20)}} - 1$$

Putting the fractions over a common denominator results in

$$x(t) = \frac{3}{5 - 4\cos(\pi t + \pi/20)}$$

Discrete-time periodic signals: The discrete-time Fourier series

Discrete-time periodic signals are represented by the discrete-time Fourier series (DTFS). We may write the DTFS of a signal x[n] with fundamental period N and fundamental frequency $\Omega_0 = 2\pi/N$, as

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n}$$

Where

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}$$

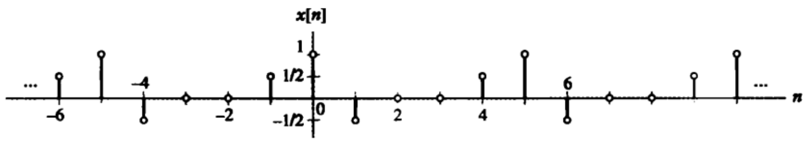
Are the DTFS coefficient of the signal x[n].We say that *x*[n] and X[k] are a DTFS pair and denote this relationship as

$$x[n] \xleftrightarrow{DTFS;\,\Omega_o} X[k]$$

The DTFS coefficients are known as the *frequency-domain representation of* x[n], because each DTFS coefficient is associated with a complex sinusoid of different frequency.

Direct calculation of DTFS coefficients:

Example: Determine the DTFS coefficient for signal x[n] shown



Solution: The signal has a period $N = 5$, so $\Omega 0 = 2\pi/5$. As the signal also has odd symmetry, it can be sum over n =-2 to n =2 in the equation and we get,

$$X[k] = \frac{1}{5}\sum_{n=-2}^{2} x[n]e^{-jk2\pi n/5}$$
$$= \frac{1}{5}\{x[-2]e^{jk4\pi/5} + x[-1]e^{jk2\pi/5} + x[0]e^{j0} + x[1]e^{-jk2\pi/5} + x[2]e^{-jk4\pi/5}\}$$

Using the values of $x[n]$, we get

$$X[k] = \frac{1}{5}\left\{1 + \frac{1}{2}e^{jk2\pi/5} - \frac{1}{2}e^{-jk2\pi/5}\right\}$$
$$= \frac{1}{5}\{1 + j\sin(k2\pi/5)\}.$$

From the above equation, we identify one period of DTFS coefficients $X[k]$, k=2 to k=-2, in rectangular and polar coordinates as

$$X[-2] = \frac{1}{5} - j\frac{\sin(4\pi/5)}{5} = 0.232e^{-j0.531}$$
$$X[-1] = \frac{1}{5} - j\frac{\sin(2\pi/5)}{5} = 0.276e^{-j0.760}$$
$$X[0] = \frac{1}{5} = 0.2e^{j0}$$
$$X[1] = \frac{1}{5} + j\frac{\sin(2\pi/5)}{5} = 0.276e^{j0.760}$$
$$X[2] = \frac{1}{5} + j\frac{\sin(4\pi/5)}{5} = 0.232e^{j0.531}.$$

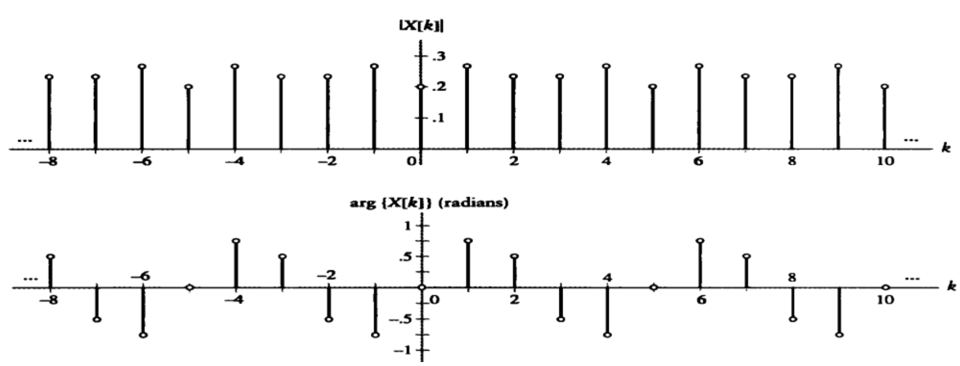


Fig: Magnitude and Phase Response of $X[k]$

The above figure shows the magnitude and phase of $X[k]$ as functions of the frequency index k.

Now suppose we calculate $X[k]$ using n = 0 to n =4 for the limits on the sum in eqn.of $X[k]$ to obtain

$$X[k] = \frac{1}{5}\{x[0]e^{j0} + x[1]e^{-j2\pi/5} + x[2]e^{-jk4\pi/5} + x[3]e^{-jk6\pi/5} + x[4]e^{-j8\pi\ 5}\}$$
$$= \frac{1}{5}\left\{1 - \frac{1}{2}e^{-jk2\pi/5} + \frac{1}{2}e^{-jk8\pi/5}\right\}.$$

Calculation of DTFS coefficients by inspection:

Example: Determine the DTFS coefficients of the signal

$$x[n] = \cos(\pi n/3 + \varphi)$$

Solution: Time period $N=6$.We expand the cosine by using Euler's formula as

$$x[n] = \frac{e^{j\left(\frac{\pi}{3}n+\phi\right)} + e^{-j\left(\frac{\pi}{3}n+\phi\right)}}{2}$$
$$= \frac{1}{2}e^{-j\phi}e^{-j\frac{\pi}{3}n} + \frac{1}{2}e^{j\phi}e^{j\frac{\pi}{3}n}.$$

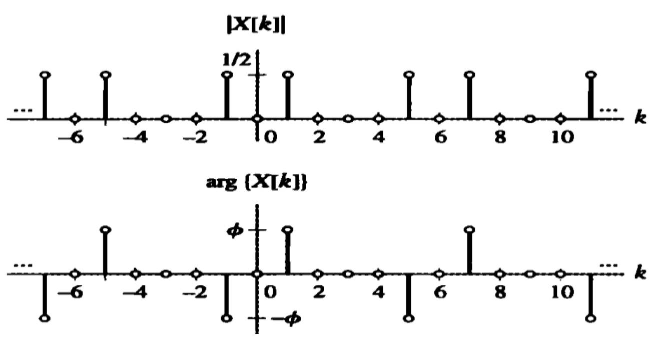
Now comparing with the DTFS equation with Ω_0= 2π/6=π/3, written by summing from k= -2 to 3

$$x[n] = \sum_{k=-2}^{3} X[k]e^{jk\pi n/3}$$
$$= X[-2]e^{-j2\pi n/3} + X[-1]e^{-j\pi n/3} + X[0] + X[1]e^{j\pi n/3} + X[2]e^{j2\pi n/3} + X[3]e^{j\pi n}.$$

Equating the terms, we get

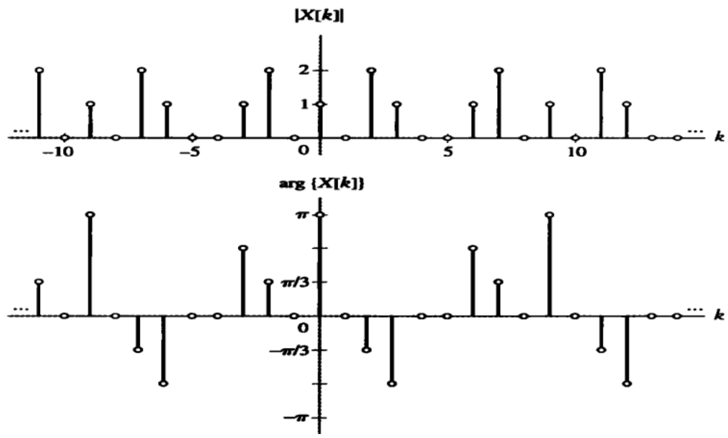
$$x[n] \xleftrightarrow{DTFS;\frac{\pi}{3}} X[k] = \begin{cases} e^{-j\phi}/2, & k = -1 \\ e^{j\phi}/2, & k = 1 \\ 0, & \text{otherwise on } -2 \le k \le 3 \end{cases}.$$

The magnitude and phase spectrum is given below



The Inverse DTFS:

Example: Determine the time signal x[n] from the DTFS coefficients given in figure below



Solution: The DTFS coefficients have period = 9, hence $\Omega_0 = 2\pi/9$. It is convenient to evaluate *x[n]* over the interval *k* = -4 to *k* = 4 to obtain

$$\begin{aligned} x[n] &= \sum_{k=-4}^{4} X[k] e^{jk2\pi n/9} \\ &= e^{j2\pi/3} e^{-j6\pi n/9} + 2e^{j\pi/3} e^{-j4\pi n/9} - 1 + 2e^{-j\pi/3} e^{j4\pi n/9} + e^{-j2\pi/3} e^{j6\pi n/9} \\ &= 2\cos(6\pi n/9 - 2\pi/3) + 4\cos(4\pi n/9 - \pi/3) - 1. \end{aligned}$$

Continuous-time nonperiodic signals: The Fourier transform

The Fourier transform is used to represent a continuous-time nonperiodic signal as a superposition of complex sinusoids. We know that the continuous nonperiodic nature of a time signal implies that the superposition of complex sinusoids used in the Fourier representation of the signal involves a continuum of frequencies ranging from -∞ to ∞. So the FT representation of a continuous-time signal involves an intyegral over the entire frequency interval; i.e.,

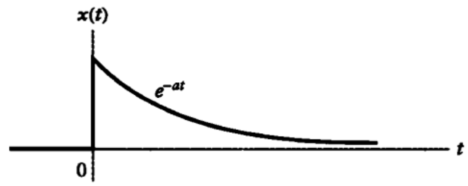
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

Where,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

Example1. FT of a real decaying exponential:

Find the FT of $x(t) = e^{-at}u(t)$ shown in the figure below.



Solution: The FT does not converge for $a \le 0$, since $x(t)$ is not absolutely integrable, i.e.;

$$\int_0^{\infty} e^{-at}\, dt = \infty \ , \quad a \le 0$$

For a > 0, we have

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\, dt \\ &= \int_0^{\infty} e^{-(a+j\omega)t}\, dt \\ &= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} \\ &= \frac{1}{a+j\omega}. \end{aligned}$$

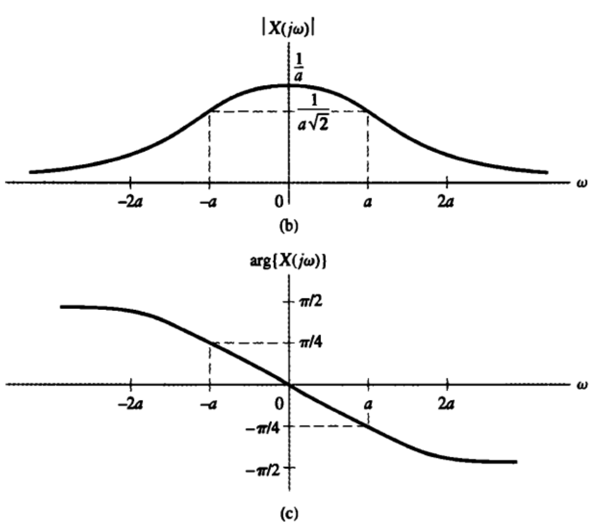
Converting to polar form, the magnitude and phase of $X(j\omega)$ are respectively given by

$$|X(j\omega)| = \frac{1}{(a^2+\omega^2)^{\frac{1}{2}}}$$

and

$$\arg\{X(j\omega)\} = -\arctan(\omega/a)$$

as shown in figure below.

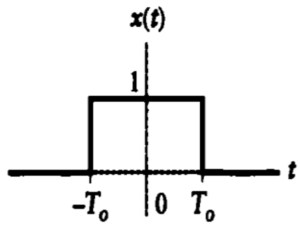


The magnitude of $X(j\omega)$ plotted against ω is known as the magnitude spectrum of the signal $x(t)$, and the phase of $X(j\omega)$ plotted as a function of ω is known as the phase spectrum of $x(t)$.

***Example* 2:   FT of a rectangular pulse**:

Consider the rectangular pulse shown figure below and defined as

$$x(t) = \begin{cases} 1, & -T_o < t < T_o \\ 0, & |t| > T_o \end{cases}.$$



Find the FT of $x(t)$.

Solution: The rectangular pulse $x(t)$ is absolutely integrable, provided that $T_0 < \infty$. So we have

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\ &= \int_{-T_o}^{T_o} e^{-j\omega t}\,dt \\ &= -\frac{1}{j\omega}e^{-j\omega t}\Big|_{-T_o}^{T_o} \\ &= \frac{2}{\omega}\sin(\omega T_o), \qquad \omega \neq 0 \end{aligned}$$

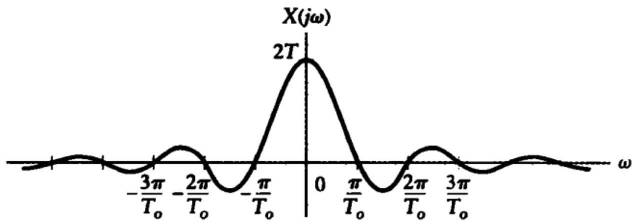
For $\omega = 0$, the integral simplifies to $2T_0$. L'Hospital's rule straightforwardly shows that

$$\lim_{\omega \to 0} \frac{2}{\omega} \sin(\omega T_0) = 2T_0$$

Thus, we usually write

$$X(j\omega) = \frac{2}{\omega} \sin(\omega\, T_0)$$

With the understanding that the value at ω = 0 is obtained by evaluating a limit. In this case $X(j\omega)$ is real and is shown in the figure below.



The magnitude spectrum is

$$|X(j\omega)| = 2\left|\frac{\sin(\omega T_o)}{\omega}\right|$$

and the phase spectrum is

$$\arg\{X(j\omega)\} = \begin{cases} 0, & \sin(\omega T_o)/\omega > 0 \\ \pi, & \sin(\omega T_o)/\omega < 0 \end{cases}$$

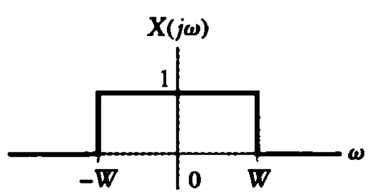
Using sinc function notation, we may write $X(j\omega)$ as

$$X(j\omega) = 2T_o \operatorname{sinc}(\omega T_o/\pi)$$

Inverse FT of a rectangular spectrum:

Example: Find the inverse FT of the rectangular spectrum (figure below) given by

$$X(j\omega) = \begin{cases} 1, & -W < \omega < W \\ 0, & |\omega| > W \end{cases}$$



Solution: Using equation for $x(t)$ for inverse FT yields

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega \\ &= \frac{1}{2j\pi t} e^{j\omega t}\Big|_{-W}^{W} \\ &= \frac{1}{\pi t}\sin(Wt), \qquad t \neq 0 \end{aligned}$$

When $t = 0$, the integral simplifies to W/π. As

$$\lim_{t \to 0} \frac{1}{\pi t}\sin(Wt) = W/\pi,$$

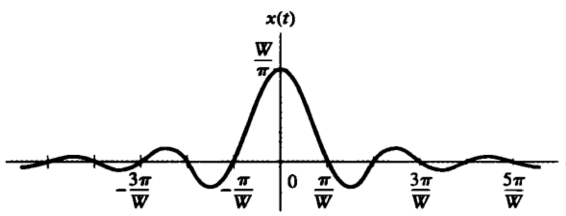
We usually write

$$x(t) = \frac{1}{\pi t}\sin(Wt)$$

or

$$x(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$

The value at $t = 0$ is obtained as a limit. The $x(t)$ is shown in the following diagram.



Discrete-time nonperiodic signals: The Discrete-time Fourier transform

The DTFT is used to represent a discrete-time nonperiodic signal as a super position of complex sinusoids. As reasoned previously, the DTFT would involve a continuum of frequencies on the interval $-\pi < \Omega < \pi$, where Ω has unit of radians. Thus, the DTFT representation of a time-domain signal involves an integral over frequency, namely,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega.$$

Where

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

is the DTFT of the signal $x[n]$. As $X(e^{j\Omega})$ and $x[n]$ are a DTFT pair, we can write

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}).$$

The transform $X(e^{j\Omega})$ describes the signal $x[n]$ as a function of a sinusoidal frequency Ω and is called the *frequency-domain* representation of $x[n]$. The equation for $x[n]$ is usually called the *inverse DTFT,* as it maps the frequency-domain representation back into the time-domain.

If

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

i.e.; if $x[n]$ is absolutely summable, then the sum in eqn. $X(e^{j\Omega})$ converges uniformly to a continuous function of ω.

Example: Find the DTFT of the sequence $x[n] = \alpha^n u[n]$.

Solution: Using the above equation, we have

$$\begin{aligned} X(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n}. \end{aligned}$$

This sum diverges for |α| ≥ 1. For |α| ≤ 1, we have the convergent geometric series

$$\begin{aligned} X(e^{j\Omega}) &= \sum_{n=0}^{\infty} (\alpha e^{-j\Omega})^n \\ &= \frac{1}{1 - \alpha e^{-j\Omega}}, \qquad |\alpha| < 1 \end{aligned}$$

If α is real valued, the denominator of the above equation may be expanded. Using Euler's formula, we get

$$X(e^{j\Omega}) = \frac{1}{1 - \alpha\cos\Omega + j\alpha\sin\Omega}$$

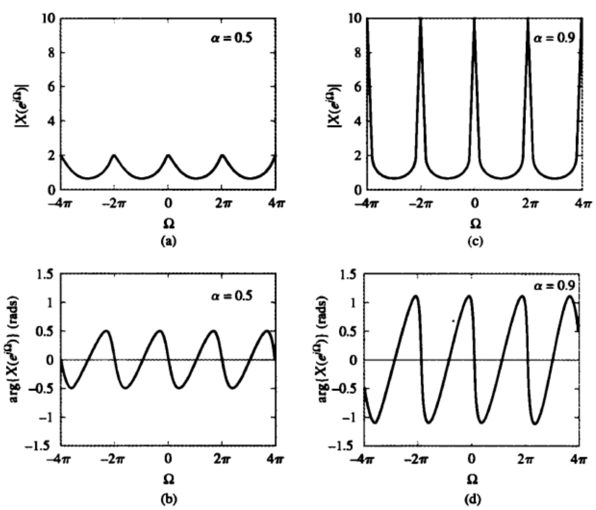
From this form, we see that the magnitude and phase spectra are given by

$$\begin{aligned} |X(e^{j\Omega})| &= \frac{1}{((1 - \alpha\cos\Omega)^2 + \alpha^2\sin^2\Omega)^{1/2}} \\ &= \frac{1}{(\alpha^2 + 1 - 2\alpha\cos\Omega)^{1/2}} \end{aligned}$$

and

$$\arg\{X(e^{j\Omega})\} = -\arctan\left(\frac{\alpha\sin\Omega}{1 - \alpha\cos\Omega}\right)$$, respectively.

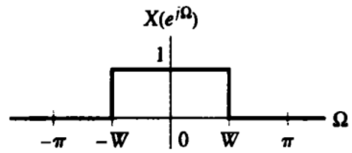
The magnitude and phase spectra for α = 0.5 and α = 0.9 are shown in the figure below. The magnitude is given and the phase is odd and both are 2π periodic.



Inverse DTFT

Example: Find the Inverse DTFT of the following rectangular spectrum

$$X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < W \\ 0, & W < |\Omega| < \pi \end{cases}$$



***Solution**:* Substituting $X(e^{j\Omega})$ in DTFT representation, we get

$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}\, d\Omega$$
$$= \frac{1}{2\pi n j} e^{j\Omega n}\Big|_{-W}^{W}, \quad n \neq 0$$
$$= \frac{1}{\pi n}\sin(Wn), \quad n \neq 0.$$

For $n = 0$, the integrand is unity and we have $x[0]=W/\pi$. Using L'Hopital's rule, we can easily show that

$$\lim_{n\to 0} \frac{1}{\pi n}\sin(Wn) = \frac{W}{\pi}$$

And thus we usually write

$$x[n] = \frac{1}{\pi n}\sin(Wn)$$

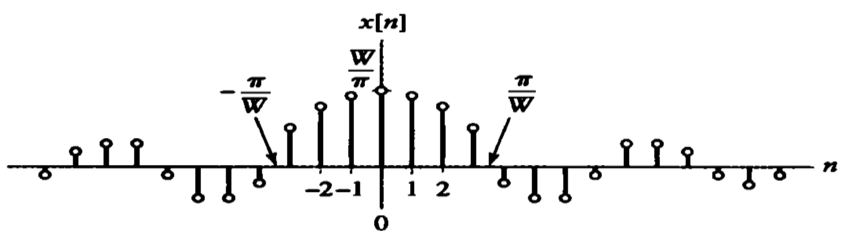


Fig: Inverse DTFT in Time domain

Properties of Fourier representation:

1. ***Linearity:***

All four Fourier representations satisfy the linearity property.

$$z(t) = ax(t) + by(t) \quad \xleftrightarrow{FT} \quad Z(j\omega) = aX(j\omega) + bY(j\omega)$$
$$z(t) = ax(t) + by(t) \quad \xleftrightarrow{FS;\omega_o} \quad Z[k] = aX[k] + bY[k]$$
$$z[n] = ax[n] + by[n] \quad \xleftrightarrow{DTFT} \quad Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$
$$z[n] = ax[n] + by[n] \quad \xleftrightarrow{DTFS;\Omega_o} \quad Z[k] = aX[k] + bY[k]$$

In these relationships, we assume that the upper case symbol denotes the Fourier representation of the corresponding lower case symbol.

2. ***Symmetry:***

Representation	*Real-Valued Time Signals*	*Imaginary-Valued Time Signals*
FT	$X^*(j\omega) = X(-j\omega)$	$X^*(j\omega) = -X(-j\omega)$
FS	$X^*[k] = X[-k]$	$X^*[k] = -X[-k]$
DTFT	$X^*(e^{j\Omega}) = X(e^{-j\Omega})$	$X^*(e^{j\Omega}) = -X(e^{-j\Omega})$
DTFS	$X^*[k] = X[-k]$	$X^*[k] = -X[-k]$

3. ***Convolution:***

a) Convolution of nonperiodic signal: Convolution of two signals $h(t)$ & $x(t)$ in the time domain corresponds to multiplication of their FT, $H(j\omega)$ & $X(j\omega)$ in frequency domain.

$$y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega)$$

A similar property holds for convolution of discrete time non-periodic signal .If

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) \text{ and } h[n] \xleftrightarrow{DTFT} H(e^{j\Omega}),\text{then}$$

$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega})\, H(e^{j\Omega})$$

b) Convolution of periodic signal: The periodic convolution of two periodic CT signals x(t) and z(t), each having period T , as

$$\begin{aligned} y(t) &= x(t) \circledast z(t) \\ &= \int_0^T x(\tau)z(t-\tau)\, d\tau \end{aligned}$$

Substituting the FS representation of z(t) into the convolution integral leads to the property

$$y(t) = x(t) \circledast z(t) \xleftrightarrow{FS;\frac{2\pi}{T}} Y[k] = TX[k]Z[k]$$

Similarly in DTFS

$$y[n] = x[n] \circledast z[n] \xleftrightarrow{DTFS;\frac{2\pi}{N}} Y[k] = NX[k]Z[k]$$

4) ***Differentiation and Integration:***

a) *Differentiation in time:* Differentiating a signal in time domain corresponds to multiplying its FT by jω in the frequency domain i.e;

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

b) *Differentiation in frequency*: Differentiation of FT in frequency domain corresponds to multiplication of the signal by –jt in the time domain i.e;

$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$$

Commonly used Differentiation and Integration properties:

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FS;\omega_o} jk\omega_o X[k]$$

$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$$

$$-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\Omega}X(e^{j\Omega})$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$$

5) Time Shift:

Let $z(t) = x(t-t_0)$ be a time –shifted version of x(t). The goal is to relate the FT of *z*(t) to the FT of x(t)

$$\begin{aligned} Z(j\omega) &= \int_{-\infty}^{\infty} z(t)e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty} x(t-t_o)e^{-j\omega t}\,dt. \end{aligned}$$

Putting $\tau = t - t_0$, we obtain

$$\begin{aligned} Z(j\omega) &= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_o)}\,d\tau \\ &= e^{-j\omega t_o}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau \\ &= e^{-j\omega t_o}X(j\omega). \end{aligned}$$

Time-shift properties of Fourier representation:

$$x(t-t_o) \xleftrightarrow{FT} e^{-j\omega t_o}X(j\omega)$$

$$x(t-t_o) \xleftrightarrow{FS;\omega_o} e^{-jk\omega_o t_o}X[k]$$

$$x[n-n_o] \xleftrightarrow{DTFT} e^{-j\Omega n_o}X(e^{j\Omega})$$

$$x[n-n_o] \xleftrightarrow{DTFS;\Omega_o} e^{-jk\Omega_o n_o}X[k]$$

Frequency-shift properties of Fourier representation:

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega-\gamma))$$

$$e^{jk_o\omega_o t}x(t) \xleftrightarrow{FS;\omega_o} X[k-k_o]$$

$$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X(e^{j(\Omega-\Gamma)})$$

$$e^{jk_o\Omega_o n}x[n] \xleftrightarrow{DTFS;\Omega_o} X[k-k_o]$$

Multiplication property:

For nonperiodic signal:

$$y(t) = x(t)z(t) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{2\pi}X(j\omega) * Z(j\omega)$$

$$y[n] = x[n]z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = \frac{1}{2\pi}X(e^{j\Omega}) \circledast Z(e^{j\Omega})$$

For periodic signal:

$$y(t) = x(t)z(t) \xleftrightarrow{FT;2\pi/T} Y[k] = X[k] * Z[k]$$

$$y[n] = x[n]z[n] \xleftrightarrow{DTFS;2\pi/N} Y[k] = X[k] \circledast Z[k]$$

Scaling properties:

Let us consider the effect of scaling the time variable on the frequency-domain representation of a signal. Beginning with the FT, let $z(t) = x(at)$, where a is a constant. By definition, we have

$$\begin{aligned} Z(j\omega) &= \int_{-\infty}^{\infty} z(t)e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty} x(at)e^{-j\omega t}\,dt. \end{aligned}$$

We effect the change of variable $\tau = at$ to get

$$Z(j\omega) = \begin{cases} (1/a)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}\,d\tau, & a > 0 \\ (1/a)\int_{\infty}^{-\infty} x(\tau)e^{-j(\omega/a)\tau}\,d\tau, & a < 0 \end{cases}.$$

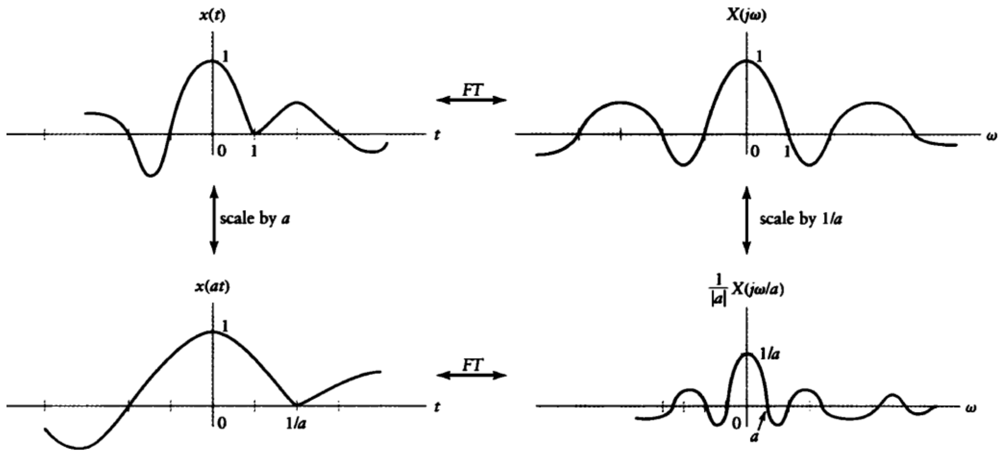
These can be combined into a single integral

$$Z(j\omega) = (1/|a|)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}\,d\tau$$

We can conclude that

$$z(t) = x(at) \xleftrightarrow{FT} (1/|a|)X(j\omega/a)$$

Scaling the signal in time introduces the inverse scaling in the frequency-domain representation and an amplitude change, as shown in the given figure:



Parseval Relationships:

It states that the energy or power in the time-domain representation of a signal is equal to the energy or power in the frequency-domain representation. So the energy and power are conserved in the Fourier representation.

The energy in a continuous-time non-periodic signal is

$$Wx = \int_{-\infty}^{\infty} |x\,(t)|^2 dt$$

Where it is assumed that $x(t)$ may be complex valued general. As $|x(t)|^2 = x(t)\ x^*(t)$, taking the conjugate of both sides of Eq, we may express $x^*(t)$ in terms of its FT X(jω) as

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}\,d\omega.$$

Substituting this formula into the expression for W_x, we obtain

$$W_x = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}\,d\omega\right]dt.$$

Now we interchange the order of integration:

$$W_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left\{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt\right\}d\omega.$$

Observing that the integral inside the braces is the FT of $x(t)$, we obtain

$$W_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)X(j\omega)\,d\omega$$

And so conclude that

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\,d\omega.$$

Hence, the energy in the time-domain representation of the signal is equal to the energy in the frequency-domain representation, normalized by 2π. The quantity $|X(j\omega)|^2$ plotted against ω is known as *energy spectrum* of the signal.

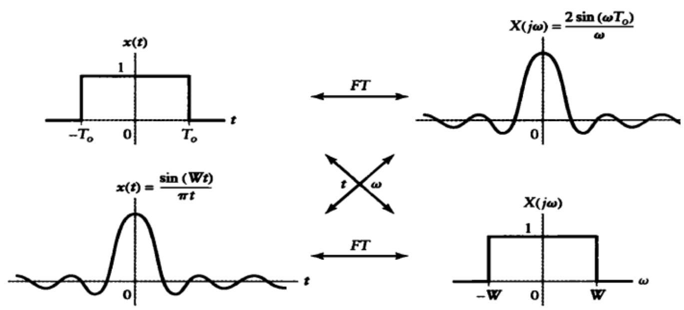
Analogous results hold for the periodic signal is known as *power density spectrum* of the signal.

The other three representations are summarized in the table below:

Representation	*Parseval Relation*
FT	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\, d\omega$
FS	$\frac{1}{T}\int_0^T \lvert x(t)\rvert^2\, dt = \sum_{k=-\infty}^{\infty} \lvert X[k]\rvert^2$
DTFT	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\Omega})\rvert^2\, d\Omega$
DTFS	$\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \sum_{k=0}^{N-1} \lvert X[k]\rvert^2$

Duality:

In this chapter, we observed a consistent symmetry between the time and frequency domain representations of signals. For example, a rectangular pulse in either time or frequency corresponds to a sinc function in either frequency or time, as shown in the figure below.



We have also observed symmetries in Fourier representation properties i.e.; convolution in one domain corresponds to modulation in other domain; differentiation in one domain corresponds to multiplication by the independent variable in the other domain, and so on. So by the consequence of this symmetry, we may interchange time and frequency. This interchangeability property is termed *duality*.

Duality properties of Fourier representation is summarized in the table below:

FT	$f(t) \xleftrightarrow{FT} F(j\omega)$	$F(jt) \xleftrightarrow{FT} 2\pi f(-\omega)$
DTFS	$x[n] \xleftrightarrow{DTFS;\,2\pi/N} X[k]$	$X[n] \xleftrightarrow{DTFS;\,2\pi/N} (1/N)x[-k]$
FS–DTFT	$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$	$X(e^{jt}) \xleftrightarrow{FS;\,1} x[-k]$

Applications of Fourier representations to mixed signal classes

- We now discuss the application of Fourier representations to mixed signals like
  - Periodic & nonperiodic signal
  - Continuous & discrete time signal
- Such mixing of signals occurs most commonly when one uses Fourier methods to
  - Analyze the interaction between signals & systems
  - Numerically evaluate properties of signal or the behavior of a system
- For example: If we apply a periodic signal to a stable LTI system, the convolution representation of the system output involves a mixing of nonperiodic (impulse response) and periodic (input) signals.
- Another example: A system that samples continuous time-signal involves both continuous & discrete-time signals.
- In order to use Fourier methods to analyze such interactions, we must build bridges between the Fourier representations of different classes of signals.
- DTFS is the only Fourier representation that can be evaluated numerically on a computer.

Fourier transform representation of periodic signals:

Neither FT nor DTFT converges for periodic signals. However, by incorporating impulses into the FT & DTFT in the appropriate manner, we may develop FT & DTFT representations of such signals.

Relating FT to FS:

The FS representation of a periodic signal x(t) is

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jkt\,\omega_0}$$

Where, ω_0 is the fundamental frequency of the signal.
As, $1 \xleftrightarrow{FT} 2\pi\delta(\omega)$, using frequency shift property, the inverse FT of a frequency shifted impulse δ (ω-$k\omega_0$) is a complex sinusoid with frequency $k\omega_0$. i.e,;

$$e^{jk\omega_o t} \xleftrightarrow{FT} 2\pi\delta(\omega - k\omega_o)$$

If we find FT of $x(t)$, then by using linearity property of FT we obtain

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t} \xleftrightarrow{FT} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_o)$$

Thus the FT of a periodic signal is a series of impulses spaced by the fundamental frequency ω_0. The k^{th} impulse has strength $2\pi X(k)$, where $X(k)$ is the k^{th} FS coefficient.
The shape of $X(j\omega)$ is identical to that of $X(k)$. The FT obtained from the FS by placing impulses at integer multiples of ω_0 and weighing them by 2π times the corresponding FS coefficient.
Given an FT consisting of impulses that are uniformly spaced in ω, we obtain the corresponding FS coefficients by dividing the impulse strengths by 2π.

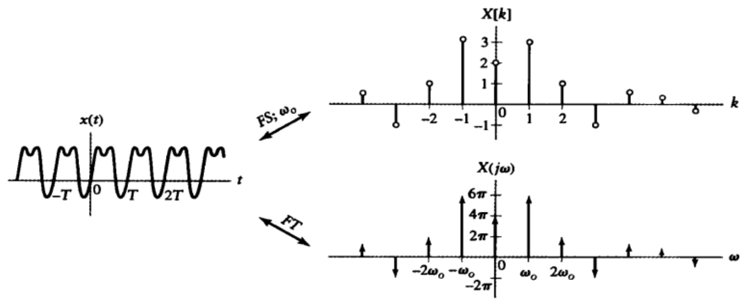


Fig: FS and FT representation of periodic continuous-time signals

Relating DTFT to DTFS:

The DTFS expression for a N periodic signal x[n] is

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n}.$$

The inverse DTFT of a frequency shifted impulse is a discrete-time complex sinusoid. The DTFT is a 2π periodic function of frequency. So we may express a frequency shifted impulse either by expressing one period such as

$$e^{jk\Omega_o n} \xleftrightarrow{DTFT} \delta(\Omega - k\Omega_o), \quad -\pi < \Omega \leq \pi, \quad -\pi < k\Omega_o \leq \pi.$$

Or, by using an infinite series of shifted impulses, separated by an interval of 2π, to obtain the 2π periodic function

$$e^{jk\Omega_o n} \xleftrightarrow{DTFT} \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_o - m2\pi)$$

The inverse DTFT equation is evaluated by means of the shifting property of the impulse function. We have

$$\frac{1}{2\pi} e^{jk\Omega_o n} \xleftrightarrow{DTFT} \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_o - m2\pi).$$

Hence we identify the complex sinusoid and the frequency shifted impulse as a DTFT pair using linearity property and substituting the above equation in equation of x[n], we get DTFT of periodic signal x[n]

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n} \xleftrightarrow{DTFT} X(e^{j\Omega}) = 2\pi \sum_{k=0}^{N-1} X[k] \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_o - m2\pi).$$

Since DTFT is 2π periodic, it follows that, DTFT of x[n] consists of a set of N impulses of strength $2\pi X(k), k=0,1,2\ldots\ldots,N-1$.

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n} \xleftrightarrow{DTFT} X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\Omega - k\Omega_o)$$

Given the DTFS coefficient and the fundamental frequency Ω_0, we obtain the DTFT representation by placing impulses at integer multiples of Ω_0 and weighting them by 2π times the corresponding DTFS coefficient.

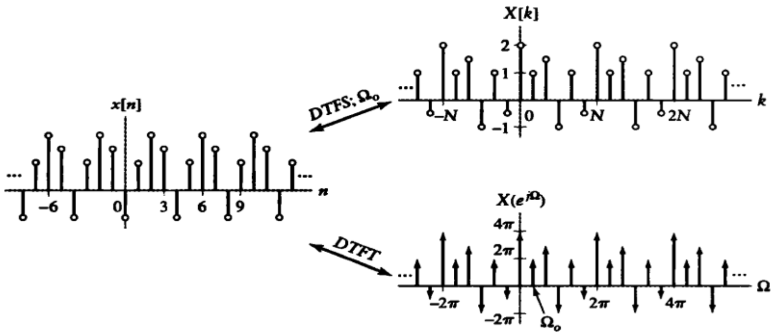


Fig: DTFS & DTFT representation of a periodic discrete-time signal

Fourier transform representation of Discrete-time signals:

In this section we derive, an FT representation of discrete-time signal by incorporating impulses into the description of the signal the appropriate manner, establishing a correspondence between the continuous frequency ω and the discrete-time frequency Ω.
Let us define the complex sinusoids

$$x(t) = e^{j\omega t} \text{ and } g[n] = e^{j\Omega n}$$

Suppose a force $g[n]$ equal to the samples of $x(t)$ taken at intervals of T_s i.e.;

$$g[n] = x(nT_s)$$

$$\Rightarrow e^{j\Omega n} = e^{j\omega T_s n}$$

From which we conclude that $\Omega = \omega T_s$.

Relating FT to DTFT:

DTFT of an arbitrary discrete-time signal $x[n]$ is

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}.$$

We want to find out an FT pair $x_\delta(t) \xleftrightarrow{FT} X_\delta(j\omega)$ that corresponds to the DTFT pair $x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$. Substituting $\Omega = \omega T_s$, we obtain the following function of continuous time frequency ω.

$$X_\delta(j\omega) = X(e^{j\Omega})\big|_{\Omega=\omega T_s},$$
$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n}.$$

Taking the inverse FT of $X_\delta(j\omega)$, using linearity and the FT pair

$$\delta(t - nT_s) \xleftrightarrow{FT} e^{-j\omega T_s n};$$

yields the continuous time signal description

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

Hence,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \xleftrightarrow{FT} X_\delta(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n}$$

The discrete-signal has values $x[n]$, while the corresponding continuous time signal consists of a series of impulses separated by T_s, with n^{th} impulse having strength $x[n]$. The DTFT $X(e^{j\Omega})$ is 2π periodic in Ω, while the FT $X_\delta(j\omega)$ is $2\pi / T_s$ periodic in ω.

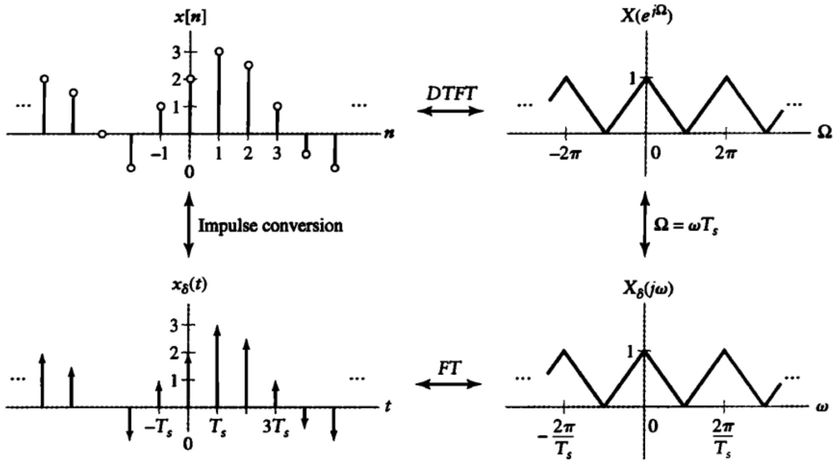


Fig: Relationship between FT and DTFT representation of a discrete-time signal

Relating FT to DTFS:

The DTFT representation of an N periodic signal $x[n]$ is given as

$$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_o)$$

Where $X[k]$ is DTFS coefficient.
Substituting $\Omega = \omega T_s$ into this eqn. yields the FT representation

$$\begin{aligned} X_\delta(j\omega) &= X(e^{j\omega T_s}) \\ &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega T_s - k\Omega_o) \\ &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(T_s(\omega - k\Omega_o/T_s)) \end{aligned}$$

Using scaling property of impulse

$$\delta(a\nu) = (1/a)\delta(\nu)$$

We can write

$$X_\delta(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\Omega_o/T_s)$$

DTFS coefficients $X[k]$ are N-periodic function, which implies that $X_\delta(j\omega)$ is periodic with period $N\Omega_0 / T_s = 2\pi / T_s$.
Continuous time representation of discrete-time signal as derived in previous chapter

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

As $x[n]$ is N periodic, so $x_\delta(t)$ is also periodic with fundamental period NT_s

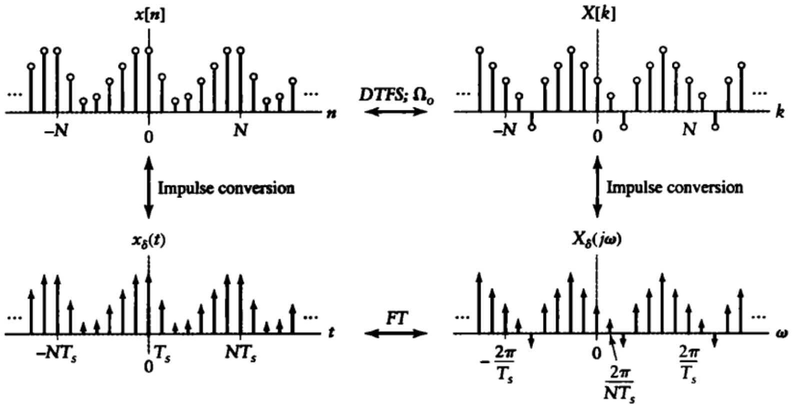


Fig: Relationship between FT and DTFS representation of a discrete-time signal

Sampling:

- The sampling operation generates a discrete-time signal from the continuous-time signal in order to manipulate the signal on a computer or microprocessor.
- Such manipulations are common in communication, control and signal processing systems.
- Sampling is also frequently performed on discrete-time signal to change the effective data rate, an operation termed *subsampling*.

Sampling continuous-time signals:

- Let $x(t)$ be a continuous-time signal. To define a discrete-time signal $x[n]$ which is equal to the *samples* of $x(t)$ at integer multiples of a sampling interval T_s, i.e.;

$$x[n] = x[nT_s]$$

- The impact of sampling is elevated by relating the DTFT of $x[n]$ to the FT of $x(t)$.
- The continuous-time representation of discrete-time signal $x[n]$ is given by

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

- Substituting $x[nT_s]$ for $x[n]$ in above eqn. we get

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Since

$$x(t)\delta(t - nT_s) = x(nT_s)\delta(t - nT_s)$$

So we may write

$$x_\delta(t) = x(t)p(t)$$

Where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

- The above eqn. implies that we may mathematically represent the sample signal as the product of original continuous-time signal and impulse train.

- This representation is commonly termed as *impulse sampling* and is a mathematical tool used only to analyze sampling.
- The effect of sampling is determined by relating FT of $x_\delta(t)$.to FT of $x(t)$.
- As multiplication in the time domain corresponds to the convolution in the frequency domain, so by multiplication property:

$$\boldsymbol{X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)}$$

- As impulse train is continuous periodic function, so its FS is given as

$$P[k] = \frac{1}{T_s}\int_{-T_{s/2}}^{T_{s/2}} \delta(t)e^{-jk\omega_0 t}\ dt = \frac{1}{T_s}$$

By using FT representation of FS

$$P(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} P[k]\delta(\omega - k\omega_s)$$

We get FT of impulse train as

$$P(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

Where $\omega_s = 2\pi/T_s$ is the sampling frequency.

$$\boldsymbol{X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)}$$

$$\boldsymbol{X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)}$$

Now we convolve $X(j\omega)$ with each of the frequency shifted impulse to get

$$\boldsymbol{X_\delta(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))}$$

- The FT of the sampled signal is given by an infinite sum of shifted versions of the original signal's FT.
- The shifted versions are offset by integer multiples of ω_s.
- The shifted versions of $X(j\omega)$ may overlap with each other if ω_s is not large enough compared with the frequency extent or bandwidth of $X(j\omega)$.
- Let the frequency component of the signal x(t) is assumed to lie within the frequency band $-W < \omega < W$ for the purpose of illustration.

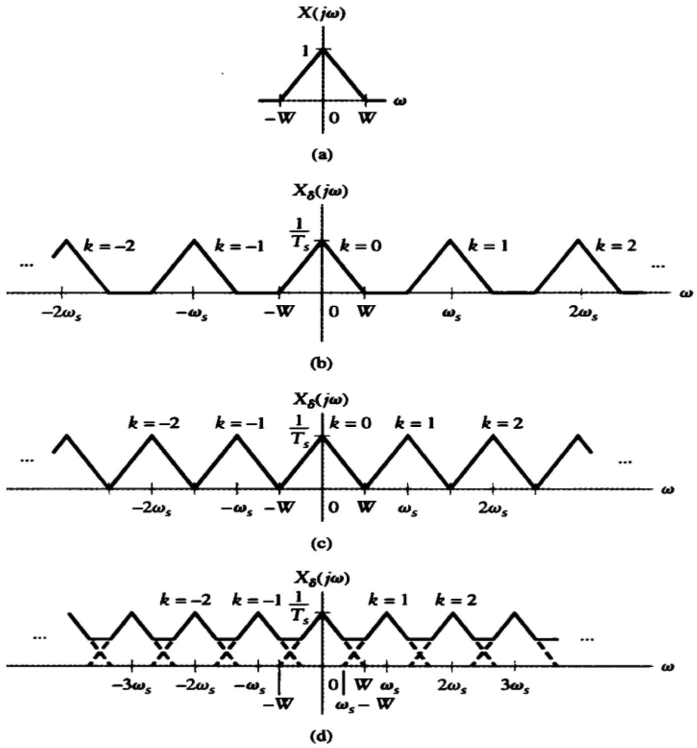


Fig: The FT of a sampled signal for different sampling frequencies (a) Spectrum of a continuous-time signal, (b) Spectrum of sampled signal when $\omega_s = 3W$, (c) Spectrum of sampled signal when $\omega_s = 2W$, (d) Spectrum of sampled signal when $\omega_s = 1.5W$

- Note that, as T_s increases and ω_s decreases, the shifted replicas of $X(j\omega)$ move close together, finally overlapping one another when $\omega_s < 2W$.
- Overlap in the shifted replicas of the original spectrum is termed *aliasing*.
- Aliasing distorts the spectrum of the sampled signal.
- The spectrum of the sampled signal no longer has a one to one correspondence with that of the original continuous-time signal.
- This means that we cannot use the spectrum of the sampled signal to analyze the continuous-time signal and we cannot uniquely reconstruct the original continuous-time signal from its samples.
- The DTFT of the sampled signal is obtained from $X_\delta(j\omega)$ by using the relationship $\Omega = \omega T_s$

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = X_\delta(j\omega)\big|_{\omega=\frac{\Omega}{T_s}}$$

- The scaling of the independent variable implies that $\omega = \omega_s$ corresponds to $\Omega = 2\pi$.
- The FTs have period ω_s, while DTFTs have period 2π.

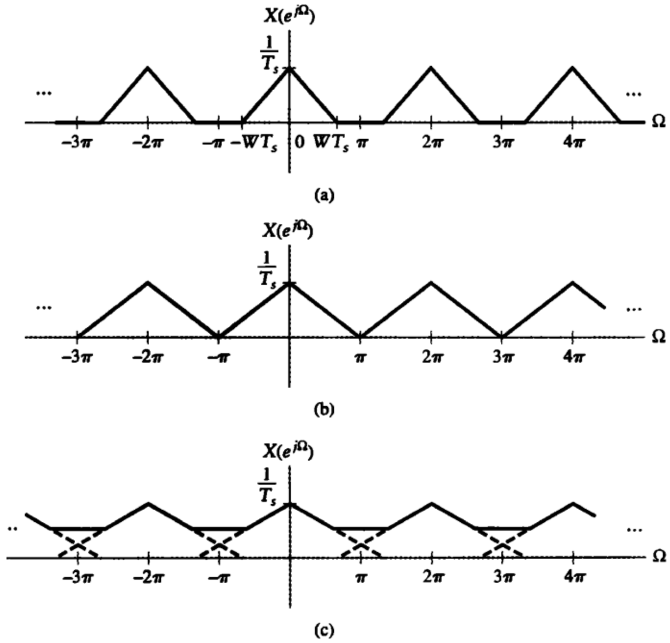
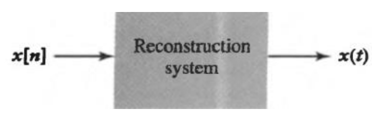


Fig: The DTFTs corresponding to the FTs depicted in Fig:1((b)-(d)). (a) ω_s = 3W, (b) ω_s = 2W, (c) ω_s = 1.5W

Reconstruction of continuous-time signals from samples:

- The problem of reconstructing a continuous-time signal from samples involving a mixture of continuous & discrete-time signals
- A device that performs reconstruction has a discrete-time input signal and a continuous-time output signal.



- The FT is well suited for analyzing this problem, since it may be used to represent both continuous & discrete-time signals.
- We first consider the conditions that must be met in order to uniquely reconstruct a continuous-time signal from its samples.

<u>*Sampling Theorem:*</u>

- *The samples of a signal do not always uniquely determine the corresponding continuous-time signal.*
- For example, the figure below shows, two different continuous-time signals having the same set of samples $x[k] = x_1(nT_s) = x_2(nT_s)$

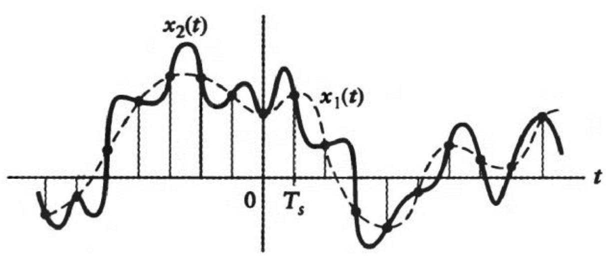


Fig: Two continuous-time signals $x_1(t)$ (dashed line) and $x_2(t)$ (solid line) that have the same set of samples

- Note that the samples do not tell us anything about the behavior of the signal in between the times it is sampled.
- In order to determine how the signal behaves in between those times, we must specify additional constraints on the continuous-time signal.
- One such set of constraints, that is very useful in practice, involves requiring the signal to make smooth transitions from one sample to another.
- The smoothness, or rate at which the time domain signal changes, is directly related to the maximum frequency that is present in the signal.
- So, constraining smoothness in the time domain corresponds to limiting the bandwidth of the signal.
- Because there is one to one correspondence between the time domain and frequency domain representations of a signal, we may also consider the problem of reconstructing the continuous-time signal in the frequency domain.
- To reconstruct a continuous-time signal uniquely from its samples, there must be a unique correspondence between the FTs of the continuous-time signal and the sampled signal.
- The FTs are uniquely related if the sampling process does not introduce aliasing.
- Aliasing distorts the spectrum of the original signal and destroys one-to-one relationship between the FTs of the continuous-time signal and the sampled signal.
- Prevention of aliasing requires satisfying the sampling theorem.
- Let $x(t) \xleftrightarrow{FT} X(j\omega)$ represent a band-limited signal so that $X(j\omega) = 0$ for $|\omega| > \omega_m$

  If $\omega_s > 2\ \omega_m$, where $\omega_s = 2\pi\ /\ T_s$ is the sampling frequency, then x(t) is uniquely determined by its samples $x(nT_s)$, n=0,±1,±2,………
- The minimum sampling frequency, 2 ω_m, is termed the *Nyquist sampling rate* or *Nyquist rate*. The actual sampling frequency, ω_s, is commonly referred to as the Nyquist frequency.
- If

  $f_m = \omega_m / 2\pi$ and $f_s > 2\ f_m$

  $=> 1/T_s > 2\ f_m$

  $=> T_s < 1/2\ f_m$

Ideal Reconstruction:

- The sampling theorem indicates how fast we must sample a signal so that the samples uniquely represent the continuous-time signal.
- If $x(t) \xleftrightarrow{FT} X(j\omega)$, then the FT representation of the sampled signal is given by:

$$X_\delta(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j\omega - jk\omega_s)$$

- The goal of reconstruction is to apply some operation to $X_\delta(j\omega)$ that converts it back to $X(j\omega)$.
- Any such operation must eliminate the replicas, images of $X(j\omega)$ that are centered at $k\omega_s$.
- This is accomplished by multiplying $X_\delta(j\omega)$ by

$$H_r(j\omega) = \begin{cases} T_s, & |\omega| \leq \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$, as shown in the figure below

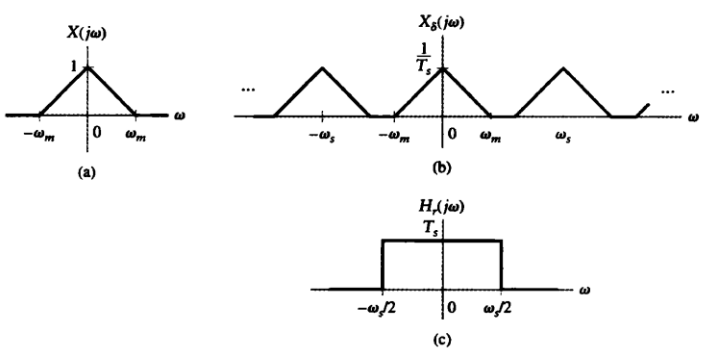


***Fig*: (a) Spectrum of original signal, (b) Spectrum of sampled signal (c) frequency response of reconstruction filter.**

- We then have $X(j\omega) = X_\delta(j\omega)\, H_r(j\omega)$
- Note that, multiplication by $H_r(j\omega)$ will not recover $X(j\omega)$ from $X_\delta(j\omega)$ if the conditions of the sampling theorem are not met and aliasing occur.
- Multiplication in the frequency domain transforms to convolution in the time domain.
- Hence, $x(t) = x_\delta(t) * h_r(t)$

  Where, $h_r(t) \xleftrightarrow{FT} H_r(j\omega)$.

  Substituting $x_\delta(t)$ in the above equation, we get

$$x(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT_s).$$

  Next we use

$$h_r(t) = \frac{T_s \sin\left(\frac{\omega_s}{2}t\right)}{\pi t}$$

$$= \text{Ts}\left(\frac{\omega_s}{2\pi}\, sinc\left(\frac{\omega_s t}{2\pi}\right)\right)$$

$$= sinc\left(\frac{\omega_s t}{2\pi}\right)$$

So,

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}(\omega_s(t - nT_s)/(2\pi))$$

- In the time domain, we construct $x(t)$ as a weighted sum of sinc functions shifted by the sampling interval. The weights correspond to the values of the discrete-tine sequence.
- The value of the $x(t)$ at $t = nT_s$ is given by $x[n]$, because all of the shifted sinc functions are zero at nT_s, except the n^{th} one and its value is unity.
- The operation described by the above equation is referred to as ideal *band limited interpolation*, since it indicates how to interpolate in between the samples of a band-limited signal.

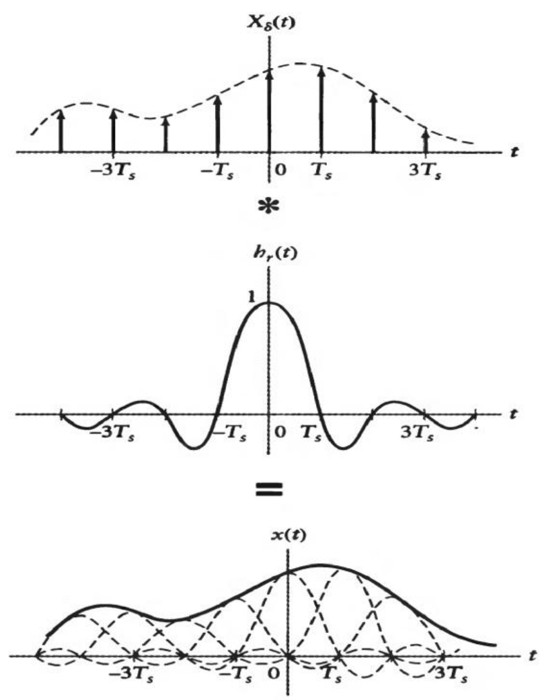


Fig: Ideal reconstruction in the time domain

Fourier series representation of finite duration on periodic signal:

- As DTFS is the only Fourier representation that can be evaluated numerically, so we apply DTFS and FS to signals that are not periodic to facilitate numerical computation of Fourier representations.
- Another advantage of this representation is understanding of relationship between the FT and corresponding FS representation.

Relating the DTFS to DTFT:

- Let $x[n]$ be a finite duration aperiodic signal of length M i.e.;

  $x[n] = 0 \quad$ for $n < 0$ and $n \geq M$

- DTFT of this signal is

$$X(e^{j\Omega}) = \sum_{n=0}^{M-1} x[n] e^{-j\Omega n}.$$

- Let $\tilde{x}[n]$ be a periodic discrete-time signal with period $N \geq M$ such that one period of $\tilde{x}[n]$ is given by $x[n]$.
- The DTFS coefficients of $\tilde{x}[n]$ are given by

$$\tilde{X}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}$$ (as $\tilde{x}[n] = x[n]$ within one period)

  Where $\Omega_0 = 2\pi / N$

$$\tilde{X}[k] = \frac{1}{N} \sum_{n=0}^{M-1} x[n] e^{-jk\Omega_o n}$$ (as $x[n] = 0$ for $n \geq M$)

  A comparison of $\tilde{X}[K]$ and $X(e^{j\omega})$ reveals that

$$\tilde{X}[k] = \frac{1}{N} X(e^{j\Omega}) \Big|_{\Omega = k\Omega_o}$$

- The DTFS coefficient of $\tilde{x}[n]$ are samples of the DTFT of $x[n]$, divided by N and evaluated by at intervals of 2π / N.
- Although $x[n]$ is not periodic, we define DTFS coefficients using $x[n]$, n = 0,1,….N-1 according to

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}$$

  So $X[K] = \tilde{X}[K] = (1/N)\, X(e^{jk\omega_0})$ (*from above two equations*)

- DTFS coefficients of $x[n]$ correspond to the DTFS coefficients of periodically extended signal $\tilde{x}[n]$.
- The effect of sampling the DTFT of a finite-duration nonperiodic is to periodically extend the signal in the time domain. i.e.;

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n + mN] \xleftrightarrow{DTFS;\Omega_o} \tilde{X}[k] = \frac{1}{N} X(e^{jk\Omega_o})$$

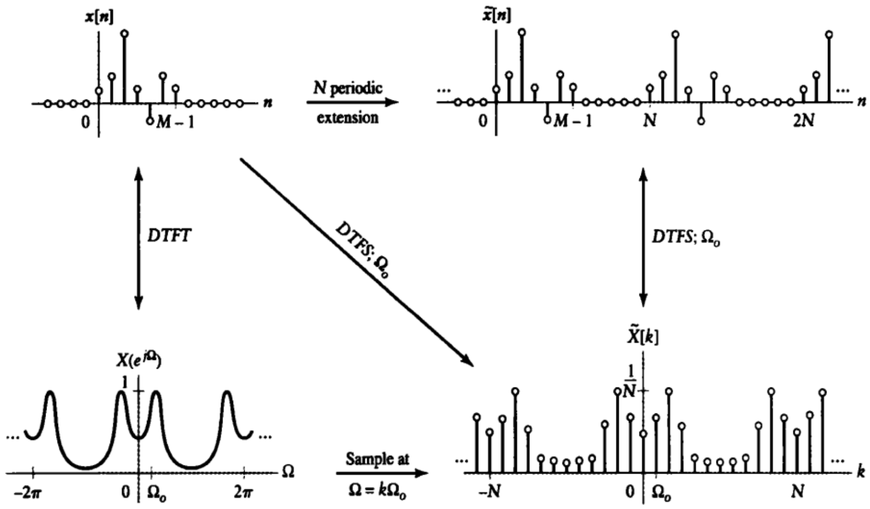


Fig: The DTFS of a finite duration nonperiodic signal

- The above relationship is dual to sampling frequency.
- Sampling a signal in time generates shifted replicas of the original signal in the frequency domain.

Dual:

- Sampling a signal in frequency generates shifted replicas of the original signal in the time domain.
- In order to prevent overlap or aliasing, of those shifted replicas in time, we require the frequency sampling interval Ω_0 to be less than or equal to $2\pi/M$ ($\Omega_0 \leq 2\pi/M \Rightarrow N \geq M$)

Relating the FS to the FT:

Let $x(t)$, an aperiodic signal have finite duration T_0,

i.e.; $x(t) = 0, \qquad t < 0 \;\&\; t \geq T_0$

Construct a periodic signal with period t

$$\tilde{x}(t) = \sum_{m=-\infty}^{\infty} x(t + mT)$$

With $T \geq T_0$ by periodically extending $x(t)$, the FS coefficients of $\tilde{x}(t)$ are

$$\tilde{X}[k] = \frac{1}{T}\int_0^T \tilde{x}(t)e^{-jk\omega_o t}\,dt$$
$$= \frac{1}{T}\int_0^{T_o} x(t)e^{-jk\omega_o t}\,dt$$

Where we have used the relationship:

$$\tilde{x}(t) = x(t) \text{ for } 0 \le t \le T_0 \text{ and } \tilde{x}(t) = 0 \text{ for } T_0 < t < T$$

The FT of $x(t)$ is defined by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

$$= \int_{0}^{T_o} x(t)e^{-j\omega t}\, dt. \quad \text{(as } x(t) \text{ is finite duration)}$$

Hence, comparing $\tilde{X}(k)$ with $X(j\omega)$

$$\tilde{X}[k] = \frac{1}{T} X(j\omega)\bigg|_{\omega = k\omega_o}$$

The FS coefficients are the samples the FT, normalized by T.

Modulation:

Modulation is basic to the operation of a communication system. Modulation provides the means for

1. Shifting the range of frequencies contained in the message signal into another frequency range suitable for transmission over the channel.

2. Performing a corresponding shift back to the original frequency range after reception of the signal.

Formally modulation is defined as the process by which some characteristic of a carrier wave is verified in accordance with the message signal. The message signal is referred to as the modulating wave, and the result of the modulation process is referred to as the modulated wave. In the receiver, demodulation is used to recover the message signal from the modulated wave. Demodulation is the inverse of modulation process.

Types of modulation:

The specific type of modulation used in a communication system is determined by the form of carrier wave used to perform the modulation. The two most commonly used forms of carrier are a sinusoidal wave and a periodic pulse train. Correspondingly, there are two classes of modulation: Continuous-wave (CW) modulation and pulse modulation.

Continuous-wave modulation:

Consider the sinusoidal carrier wave

$$c(t) = A_c \cos(\varphi(t))$$

which is uniquely defined by the carrier amplitude A_c and angle $\varphi(t)$. Depending on the type of parameter chosen for modulation, two subclasses of CW modulation is identified. i.e.,

- *Amplitude modulation*, in which the carrier amplitude is varied with the message signal.
- *Angle modulation*, in which the angle of carrier is varied with the message signal.

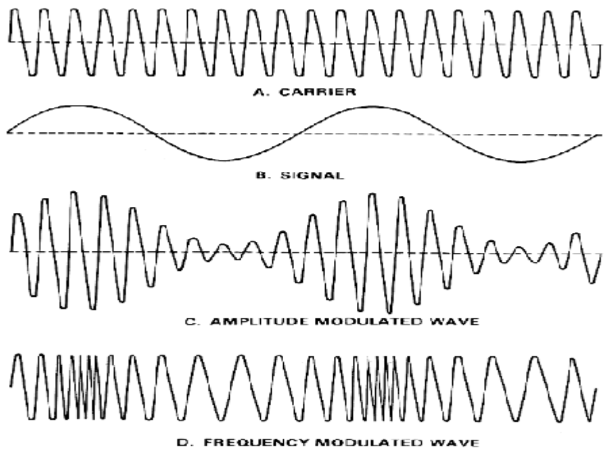


Fig: Amplitude and angle modulated signal

Pulse modulation:

Consider a carrier wave

$$c(t) = \sum_{n=-\infty}^{\infty} p(t - nT)$$

that consists of a periodic train of narrow pulses, where ‘T’ is the period and p(t) denotes a pulse of relatively short duration. When some characteristic parameter of p(t) is varied in accordance with the message signal, we get pulse modulation. Depending on how pulse modulation is actually accomplished, the two subclasses are

- *Analog pulse modulation*, in which a characteristic parameter such as amplitude, duration or position of a pulse is varied continuously with the message signal.
- *Digital pulse modulation,* in which the modulated signal is represented in coded form known as pulse code modulation.

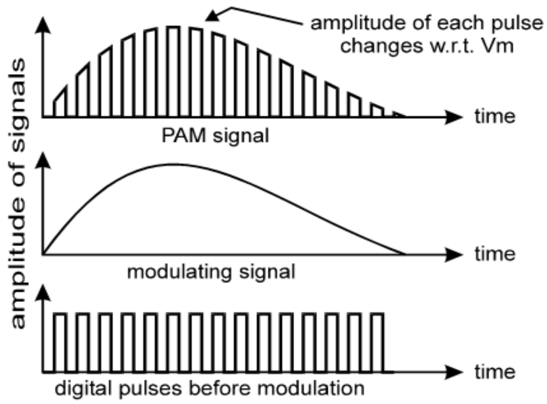


Fig: Pulse amplitude modulation waveform

Benefits of modulation

In communication system, four benefits which results from the use of modulation are:

1. Modulation is used to shift the spectral content of a message signal so that it lies inside the operating frequency band of a communication channel. It is useful for long distance and high speed transmission. E.g., The telephonic communication over a cellular radio channel, where 300-3100 Hz frequency are shifted to 800-900 MHz frequency, which is assigned to cellular radio system in North America.

2. Modulation provides a mechanism for putting the information content of a message signal into a form that may be less vulnerable to noise or interference.

3. It permits the use of multiplexing. Multiplexing permits the simultaneous transmission of information bearing signals from a number of independent sources over the channel and on to their respective destinations which makes communication channels cost-effective.

4. Modulation makes it possible for the physical size of the transmitting or receiving antenna to assume a practical value. Electromagnetic theory says that the physical aperture of an antenna is directly comparable to the wavelength of the radiated or incident electromagnetic signal. Alternatively, since wavelength and frequency are inversely related we may say that the aperture of the antenna is inversely proportional to the operating frequency. Modulation elevates the spectral content of the modulating signal by an amount equal to the carrier frequency. Hence, the larger the carrier frequency, the smaller will be the physical aperture of the transmitting as well as the receiving antenna.

Full amplitude modulation:

Let us consider a sinusoidal carrier wave

$$c(t) = A_c cos(\omega_c t).$$

For convenience of presentation, we have assumed that the phase of the carrier wave is zero in above equation (as the primary emphasis here is on variations imposed on the carrier amplitude). Let $m(t)$ represent a message signal of interest. Amplitude modulation (AM) is defined as a process in which the amplitude of the carrier is varied in proportion to a message signal $m(t)$.

$$s(t) = A_c[1+k_a m(t)]\ cos(\omega_c t). \quad \dots\dots\dots\dots\dots\dots..(1)$$

where k_a is a constant called the amplitude sensitivity factor of the modulator. The modulated wave $s(t)$ o defined is said to be a 'full' AM wave. Here, the radian frequency ω_c of the carrier is maintained constant.

Percentage of modulation: Is called the envelope of the AM wave $s(t)$. Using $a(t)$ to denote this envelope, the equation may be written as

$$a(t) = A_c|1+k_a m(t)| \dots\dots\dots\dots\dots\dots\dots.(2)$$

Two conditions arise, depending on the magnitude of $k_a m(t)$, compared with unity:

1. Undermodulation, governed by the the condition

$$|k_a m(t)| \leq 1, \quad \text{for all } t.$$

Under this condition, the term $1+k_a m(t)$ is always nonnegative. Therefore, the expression for the envelope of the AM wave may be simplified as

$$a(t) = A_c[1+k_a m(t)], \quad \text{for all } t.$$

2. Overmodulation, governed by the weaker condition

$$|k_a m(t)| > 1, \quad \text{for some } t.$$

Under this condition, the equation (*2*) is used in evaluating the envelope of the AM wave.

% modulation = The maximum absolute value of $k_a m(t) \times 100$.

Accordingly, the first condition corresponds to a percentage modulation ≤ 100%, whereas the second condition corresponds to a percentage modulation > 100%.

Generation of AM wave:

Various schemes have been devised for the generation of an AM wave. Let us consider a simple circuit that follows from the defining equation (*1*). This equation can be rewritten as :

$$s(t) = k_a\,[m(t)+B]\,A_c \cos(\omega_c t).$$

The constant B, equal to $1/\,k_a$, represents a bias that is added to the message signal $m(t)$ before modulation. The above equation suggests the scheme described in the block diagram given below for generating an AM wave. Basically it consists of two functional blocks:

- An adder that adds the bias B to the incoming message signal $m(t)$
- A multiplier that multiplies the adder output $[m(t) + B]$ by the carrier wave $A_c \cos(\omega_c t)$, producing the AM wave $s(t)$.

The percentage modulation is controlled by adjusting the bias B.

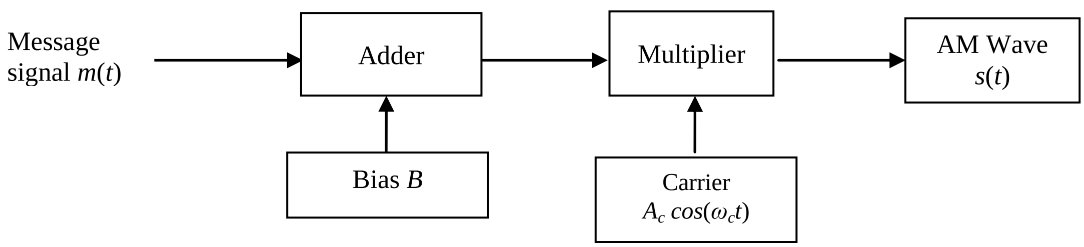


Fig: System for generating an AM wave

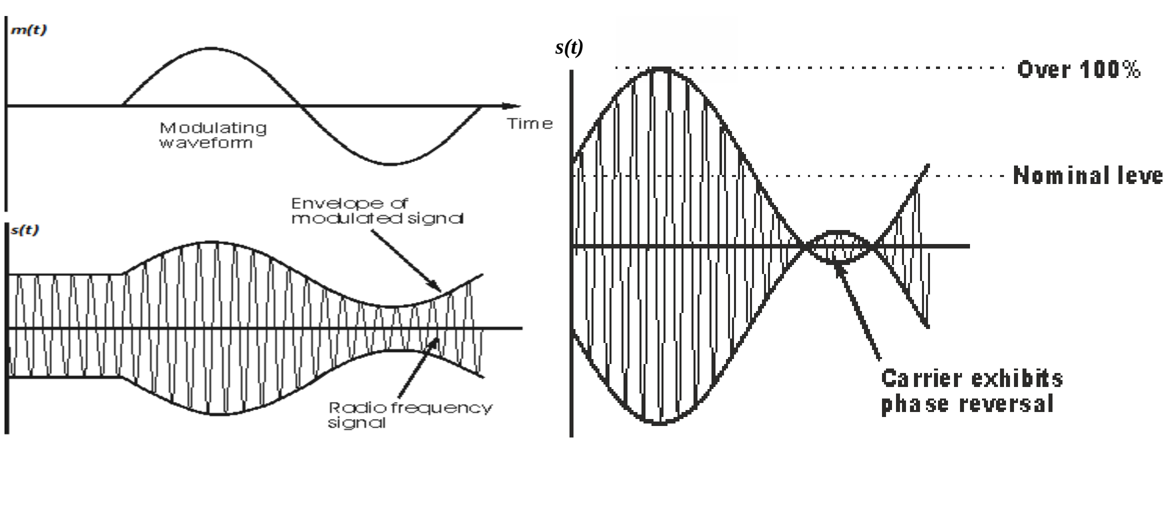


Fig: Waveform of Amplitude modulation for a varying percentage of modulation

Frequency domain description of amplitude modulation:

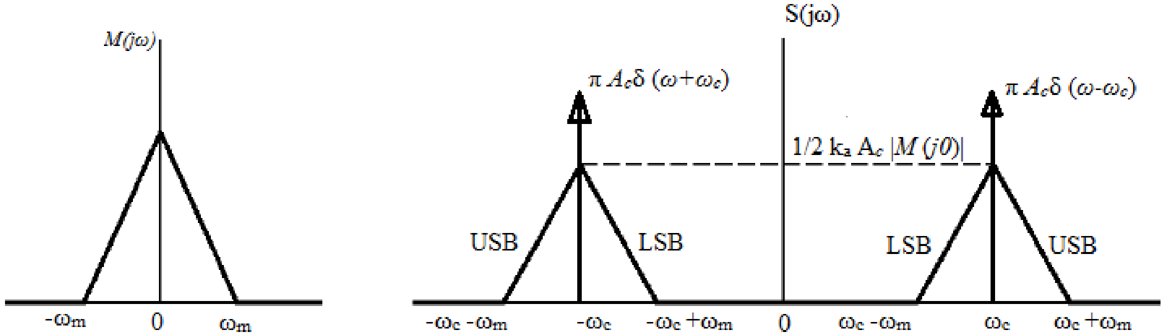
To develop the frequency description of AM wave s(t), we take the Fourier transform of both sides of equation (*1*). Let S(jω) denote Fourier transform of s(t) and M(jω) denote Fourier transform of m(t).

- The Fourier transform of $A_c \cos(\omega_c t)$ is $\pi A_c [\delta(\omega-\omega_c) + \delta(\omega+\omega_c)]$
- The Fourier transform of $m(t) \cos(\omega_c t)$ is $\frac{1}{2} [M(j\omega-j\omega_c) + M(j\omega+j\omega_c)]$

Using these results and invoking the linearity and scaling properties of the Fourier transform, the Fourier transform of the AM wave is expressed as

$$S(j\omega) = \pi A_c [\delta(\omega-\omega_c) + \delta(\omega+\omega_c)] + \frac{1}{2} k_a A_c [M(j\omega-j\omega_c) + M(j\omega+j\omega_c)] \;\ldots\ldots(3)$$

Let the message signal m(t) be band limited to the interval $-\omega_m \leq \omega \leq \omega_m$ as shown in the figure below.



(a) (b)

Fig:

We refer to the highest frequency component ω_m of $m(t)$ as the message bandwidth, measured in rad/s. We find from the eqn (*3*) that the spectrum $S(j\omega)$ of the AM wave shown in the figure (b) above for the case where $\omega_c > \omega_m$. This spectrum consists of two impulse functions weighted by the factor πA_c and occurring at $\pm \omega_c$, and two versions of the message spectrum shifted in frequency by $\pm \omega_c$ and scaled in amplitude $\frac{1}{2} k_a A_c$. The spectrum in fig (b) described as follows:

a) For positive frequencies, the portion of the spectrum of the modulated wave lying above the carrier frequency ω_c is called *upper sideband*, where as the symmetric portion below ω_c is called *lower sideband.* For negative frequencies, this condition is reversed. The condition $\omega_c > \omega_m$ is a necessary condition for the side bands not to overlap.

b) For positive frequencies, the highest frequency component of the AM wave is $\omega_c + \omega_m$ and the lowest frequency component of the AM wave is $\omega_c - \omega_m$. The difference between these two frequencies defines the *transmission bandwidth* ω_T for an AM wave which is exactly twice the *message bandwidth* ω_m, i.e., $\omega_T = 2\,\omega_m$. The spectrum of AM wave as depicted in fig (b) is *full*, in that the carrier, the upper sideband and the lower sideband are all completely represented. Hence, this form of modulation is called as *full amplitude modulation*.

Demodulation of AM Wave:

Envelope detector is used for demodulation of AM wave, shown in the figure below, which consists of a diode and a resistor-capacitor filter. The operation of this envelope detector is as follows:

On the positive half-cycle of the input signal, the diode is forward biased and the capacitor *C* charges up rapidly to the peak value of the input signal. When the input signal falls below this value the diode becomes reverse biased and the capacitor *C* discharges slowly through the load resistor R_l. The discharging process continues until the next positive half cycle. When the input signal becomes greater than the voltage across the capacitor, the diode conducts again and the process is repeated. Here it is assumed that the diode is an ideal diode, the load resistance R_l is large compared with the source resistance R_s. During the charging process, the time constant is effectively equal to RsC. This time constant must be short compared with the carrier period $2\pi/\omega_c$, i.e.,

$RsC << 2\pi/\omega_c$

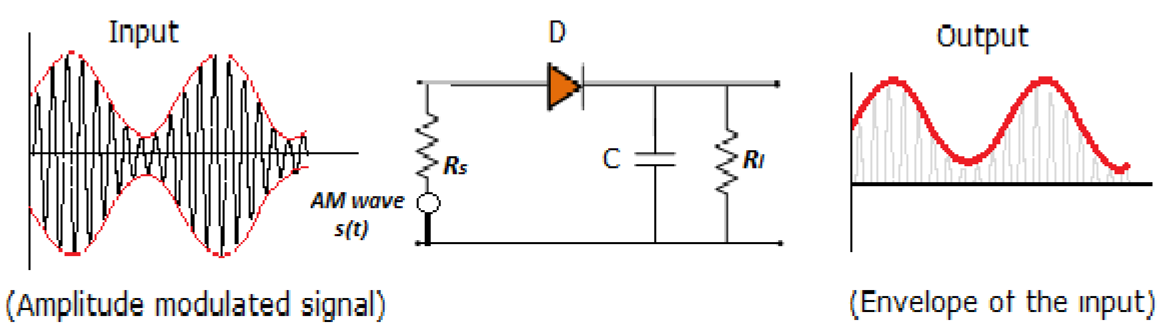


Fig: Circuit diagram of Envelope detector showing its input & output.

Accordingly the capacitor *C* charges rapidly and thereby follows the applied voltage upto the positive peak when the diode is conducting. In contrast, when the diode is reverse biased, the discharging time constant is equal to R_lC. This second time constant must be long enough to ensure that the capacitor discharges slowly through the load resistor R_l between positive peaks of the carrier wave, but not so long that the capacitor voltage will not discharge at the maximum rate of change of modulating wave, i.e.,

$$2\pi/\omega_c << R_lC << 2\pi/\omega_m.$$

Pulse Amplitude Modulation:

Pulse amplitude modulation (PAM) is a widely used form of pulse modulation. The basic operation in PAM systems is the *sampling* that includes the derivation of sampling theorem and related issues of aliasing and reconstructing the message signal from its sampled version.

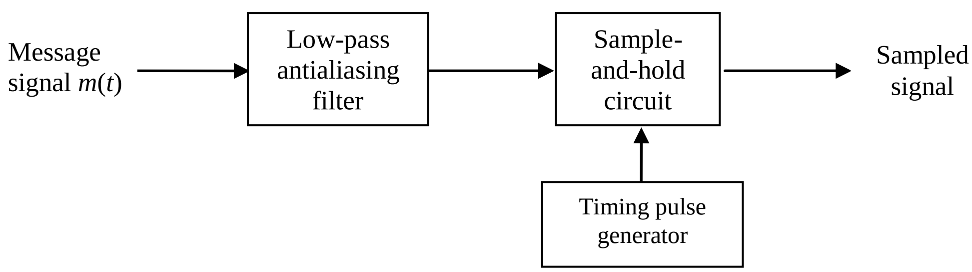


Fig: System for generating a flat-topped PAM signal

The sampling theorem in the context of PAM is in two equivalent parts as follows:

1) A band-limited signal of finite energy that has no radian frequency components higher than ω_m is uniquely determined by the values of the signal at instant of time separated by π / ω_m seconds.
2) A band-limited signal of finite energy that has no radian frequency components higher than ω_m may be completely recovered from knowledge of its samples taken at the rate of ω_m / π per seconds.

Part 1 of sampling theorem is exploited in the transmitter of a PAM system and part 2, in the receiver of the system. The special value of the sampling rate ω_m / π is referred to as the *Nyquist rate.* To combat the effect of aliasing in practice, we use two corrective measures:

- Prior to sampling, a low pass anti-aliasing filter is used to attenuate high frequency component of the signal which lie outside the band of interest.
- The filtered signal is sampled at a rate slightly higher than the *Nyquist rate.*

Mathematical description of PAM:

PAM is a form of pulse modulation, in which the amplitude of the pulsed carrier is varied in accordance with instantaneous sample values of the message signal.

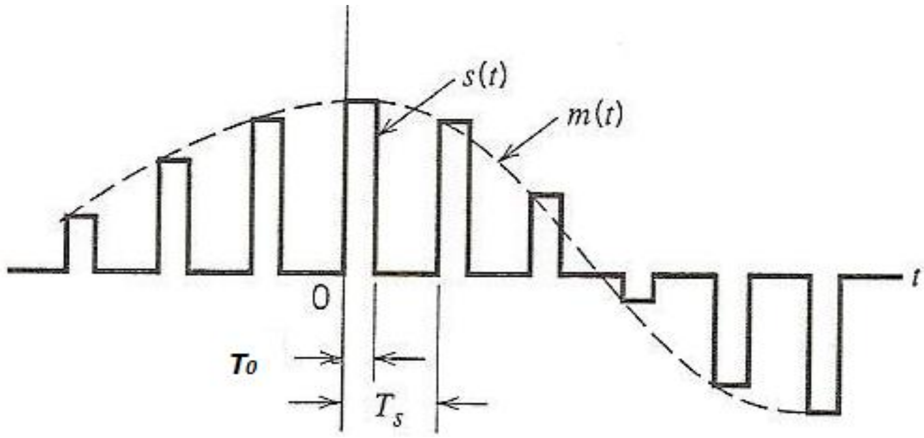


Fig: Wave form of flat topped PAM signal

For a mathematical representation of PAM signal s(t) for a message signal $m(t)$, we may write

$$s(t) = \sum_{n=-\infty}^{\infty} m[n]h(t - nT_s)$$

Where, Ts → sampling period

$m[n]$ → the value of message signal m(t) at time $t=nT_s$

$h(t)$ → a rectangular pulse of unit amplitude and duration T_0

The impulse sampled version of the message signal $m(t)$ is given by

$$m_\delta(t) = \sum_{n=-\infty}^{\infty} m[n]\delta(t - nT_s)$$

The PAM signal is expressed as

$$s(t) = \sum_{n=-\infty}^{\infty} m[n]h(t - nT_s)$$

$$= m_\delta(t) * h(t)$$

The above equation states that s(t) is mathematically equivalent to the convolution of $m_\delta(t)$ - the impulse sampled version of m(t) – and the pulse h(t).

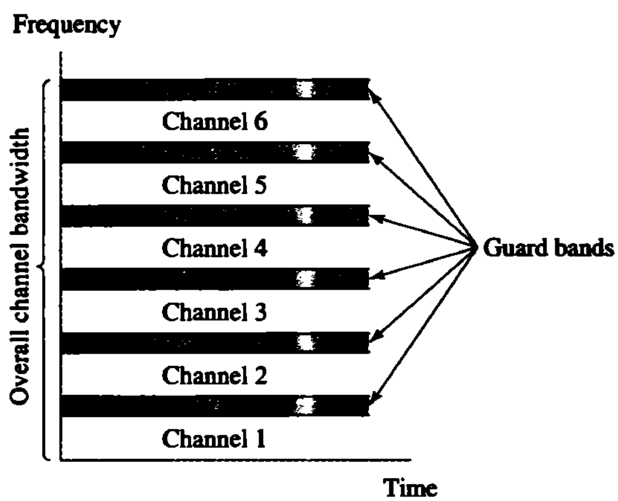
Multiplexing:

Modulation provides a method for multiplexing whereby message signal derived from independent sources are combined into a composite signal suitable for transmission over a common channel.

In telephone system, the signals from different speakers are combined in such a way that they do not interfere with each other during transmission and so that they can be separated at the receiving end.

Multiplexing can be accomplished by separating different message signals either in frequency, or time, or through the use of coding techniques. Thus, there are three basic types of multiplexing, viz:

***a) Frequency-division multiplexing*:** In this type of multiplexing, the signals are separated by allocating them to different frequency bands. FDM favours use of CW modulation, where each message signal is able to use the channel on a continuous-time basis.



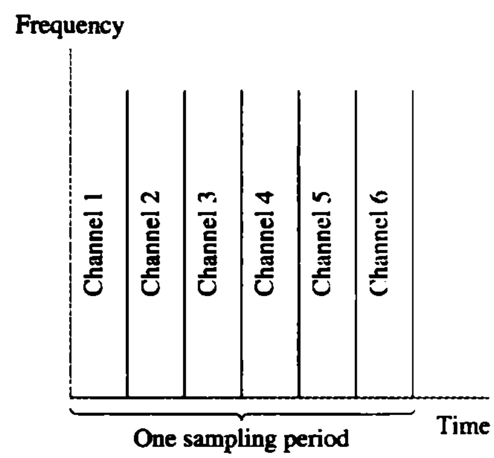


Fig: (a) Frequency-division multiplexing (b) Time-division multiplexing

***b) Time-division multiplexing*:** Here, the signals are separated by allocating them to different time slots within a sampling interval . TDM favours the use of pulse modulation, where each message signal has access to the complete frequency response of the channel.

***c) Code-division multiplexing*:** It relies on the assignment of different codes to the individual users of the channel.

a) Frequency-division multiplexing**:** The block diagram of FDM system is shown below. The low pass filters are used for band limiting the input signals. The filtered signals are applied to modulators that shift the frequency ranges of the signals so as to occupy mutually exclusive frequency intervals. The band pass filters following the modulators are used to restrict the band of each modulated wave to its prescribed range. Next, the resulting band pass filters are summed to form the input to the common channel. At the receiving terminal, a bank of band pass filters, with their inputs, connected in parallel, is used to separate the message signal on a frequency occupancy basis. Finally, the original message signals are recovered by individual demodulators.

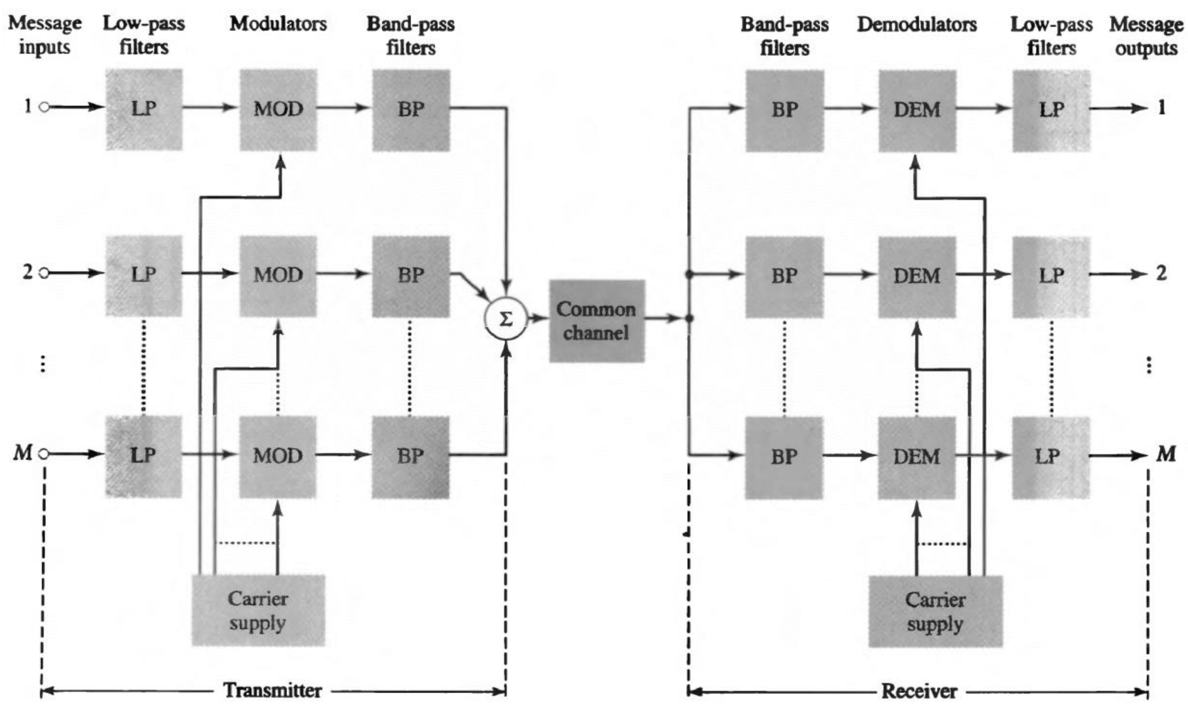


Fig: Block diagram of FDM system

b) Time-division multiplexing**:** The basic operation of a TDM system is the sampling theorem, which states that we can transmit all the information contained in a band limited message signal by using samples of the signal taken uniformly at a rate that is usually higher than the Nyquist rate. The important feature of the sampling process is conservation of time i.e., the transmission of the message samples engages the transmission channel for only a fraction of sampling interval on a periodic basis, equal to the width T_0 of a PAM modulating pulse. In this way, some of the time interval between adjacent samples is cleared for use by other independent message sources on a time shared basis.

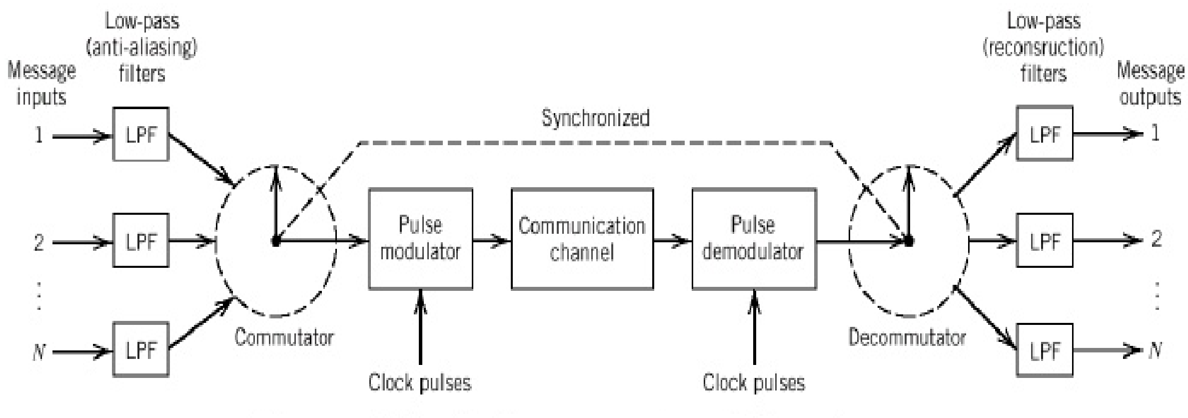


Fig: Block diagram of TDM system

The concept of TDM is illustrated by the above block diagram.

- Each input message signal is first restricted in band width by a low pass filter to remove the frequency that is non essential to an adequate representation of the signal.
- LPF output applied to a commutator that is usually implemented by means of electronic switching circuitry.
- The function of the commutator is two fold. 1) to take a narrow sample of each of the M input message signal at a rate $1/T_s$ i.e., slightly higher than ω_c / π, where ω_c is the cut off frequency of LPF. 2) to sequentially interleave these *M* samples inside a sampling interval *Ts.*
- The multiplexed signal is applied to a pulse modulator that transforms it into a form suitable for transmission over a common channel.
- At receiver, the signal is applied to a pulse demodulator which performs inverse operation of pulse modulator.
- The narrow samples produced are distributed to the appropriate low pass reconstruction filter by decommutator.
- Synchronization between timing operation of the transmitter and receiver in a TDM system is essential for satisfactory performance of the system.
- Synchronization may be achieved by inserting an extra pulse into each sampling interval on regular basis.
- The combination of *M* PAM signals and a synchronization pulse combined in a single sampling period is referred to as a *frame* synchronization.

Phase and Group delay:

Whenever a signal is transmitted a through a frequency-selective system, such as communication channel, some *delay* is introduced into the output signal in relation to the input signal. The delay is determined by the phase response of the system.

Let the phase response of a dispersive communication channel is represented by:

$\varphi(\omega) = \arg\{H(j\omega)\}$, where $H(j\omega) \rightarrow$ frequency response of the channel

Suppose that a sinusoidal signal is transmitted through the channel at a frequency ω_c .

The signal received at the channel output lags the transmitted signal by $\varphi(\omega_c)$ radians.

The time delay corresponding to this phase lag, which is known as *phase delay* (τ_p):

$$\tau_p = -\frac{\varphi(\omega_c)}{\omega_c},$$ where minus sign (-) denotes the lag.

Note: The phase delay is not necessarily the true signal delay.

Let us consider a transmitted signal

$$s(t) = A\cos(\omega_c t)\cos(\omega_0 t),$$

consisting of a DSB-Sc modulated wave with carrier frequency ω_c and sinusoidal modulation frequency ω_0.

Expressing the modulated signal $s(t)$ in terms of its upper and lower side frequencies, it may be written as:

$$s(t) = \frac{1}{2} A\cos(\omega_1 t) + \frac{1}{2} A\cos(\omega_2 t)$$

where, $\omega_1 = \omega_c + \omega_0$ and $\omega_2 = \omega_c - \omega_0$

If $\omega_0 << \omega_c$ => side frequencies ω_1 & ω_2 are close together, with ω_c between them. Such a modulated signal is called *narrowband signal*. The phase response $\varphi(\omega)$ may be approximated in the vicinity of $\omega = \omega_c$ by the two-term Taylor series expansion

$$\phi(\omega) = \phi(\omega_c) + \left.\frac{d\phi(\omega)}{d\omega}\right|_{\omega=\omega_c} \times (\omega - \omega_c).$$

The time delay incurred by the message signal (i.e., the *envelope* of the modulated signal) is given by:

$$\tau_g = -\left.\frac{d\phi(\omega)}{d\omega}\right|_{\omega=\omega_c}.$$

The time delay τ_g is called the group delay or *envelope delay*. The group delay is defined as the negative of the derivative of the phase response $\varphi(\omega)$ of the channel with respect to ω , evaluated at the carrier frequency ω_c.

Note: The time delay is a true signal delay.

For wide-band modulated signal, the frequency components of the message signal are delayed by different amounts at the channel output. Consequently, the message signal undergoes a form of linear distortion known as *delay distortion*. To reconstruct a faithful version of the original message signal in the receiver, we have to use a *delay equalizer*. This equalizer has to be designed in such a way that when it is connected in cascade with the channel, the overall group delay is constant.

Introduction to the Laplace Transform

- Fourier transforms are extremely useful in the study of many problems of practical importance involving signals and LTI systems.
- They are purely imaginary complex exponentials e^{st}, $s=j\omega$
- A large class of signals can be represented as a linear combination of complex exponentials and complex exponentials are eigenfunctions of LTI systems.
- However, the eigenfunction property applies to any complex number s, not just purely imaginary (signals).
- This leads to the development of the ***Laplace transform*** where s is an arbitrary complex number.
- Laplace and z-transforms can be applied to the analysis of un-stable system (signals with infinite energy) and play a role in the analysis of system stability
- The response of an LTI system with impulse response $h(t)$ to a complex exponential input, $x(t)=e^{st}$, is

$$y(t) = H(s)e^{st}$$

where s is a complex number and

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

when s is purely imaginary, this is the ***Fourier transform***, $H(j\omega)$
when s is complex, this is the ***Laplace transform*** of $h(t)$, $H(s)$
The Laplace transform of a general signal $x(t)$ is:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

and is usually expressed as:

$$x(t) \overset{L}{\leftrightarrow} X(s)$$

Laplace and Fourier Transform
The Fourier transform is the Laplace transform when s is purely imaginary:

$$X(s)\big|_{s=j\omega} = F\{x(t)\}$$

An alternative way of expressing this is when $s = \sigma+j\omega$

$$\begin{aligned} X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt \\ &= \int_{-\infty}^{\infty} \left[x(t)e^{-\sigma t}\right]e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x'(t)e^{-j\omega t}dt \\ &= F\{x'(t)\} \end{aligned}$$

The Laplace transform is the Fourier transform of the transformed signal $x'(t) = x(t)e^{-\sigma t}$. Depending on whether σ is positive/negative this represents a growing/negative signal

Example 1: Laplace Transform

Consider the signal $x(t)=e^{-at}u(t)$
The Fourier transform $X(j\omega)$ converges for $a>0$:

$$X(j\omega)=\int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt=\int_{0}^{\infty} e^{-at}e^{-j\omega t}dt=\frac{1}{j\omega+a},\quad a>0$$

The Laplace transform is:

$$X(s)=\int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt=\int_{0}^{\infty} e^{-(s+a)t}dt$$
$$=\int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt$$

which is the Fourier Transform of $e^{-(\sigma+a)t}u(t)$

Or $X(\sigma+j\omega)=\dfrac{1}{(\sigma+a)+j\omega},\quad \sigma+a>0$

$$e^{-at}u(t)\overset{L}{\leftrightarrow}X(s)=\frac{1}{s+a},\quad \mathrm{Re}\{s\}>-a$$

If a is negative or zero, the Laplace Transform still exists

Example 2:

Consider the signal $x(t)=-e^{-at}u(-t)$
The Laplace transform is:

$$X(s)=-\int_{-\infty}^{\infty} e^{-at}e^{-st}u(-t)dt$$
$$=-\int_{-\infty}^{0} e^{-(s+a)t}dt$$
$$=\frac{1}{s+a}$$

Convergence requires that $\mathrm{Re}\{s+a\}<0$ or $\mathrm{Re}\{s\}<-a$.
The Laplace transform expression is identical to Example 1 (similar but different signals), however the regions of convergence of s are mutually exclusive (non-intersecting).
For a Laplace transform, we need both the expression and the Region Of Convergence (ROC).

Example 3:

The Laplace transform of the signal $x(t) = \sin(\omega t)u(t)$ is:

$$X(s)=\int_{-\infty}^{\infty}\tfrac{1}{2j}\left(e^{j\omega t}-e^{-j\omega t}\right)u(t)e^{-st}dt$$
$$=\tfrac{1}{2j}\int_{0}^{\infty} e^{-(s-j\omega)t}dt-\tfrac{1}{2j}\int_{0}^{\infty} e^{-(s+j\omega)t}dt$$
$$=\tfrac{1}{2j}\left(\left.\frac{e^{-(s-j\omega)t}}{-(s-j\omega)}\right|_0^{\infty}+\left.\frac{e^{-(s+j\omega)t}}{(s+j\omega)}\right|_0^{\infty}\right)$$
$$=\tfrac{1}{2j}\left(\frac{1}{(s-j\omega)}-\frac{1}{(s+j\omega)}\right)$$
$$=\tfrac{1}{2j}\left(\frac{2j\omega}{s^2+\omega^2}\right)$$
$$=\frac{\omega}{s^2+\omega^2}$$

Fourier Transform does not Converge …

It is worthwhile reflecting that the Fourier transform does not exist for a fairly wide class of signals, such as the response of an unstable, first order system, the Fourier transform does not exist/converge

E.g. $x(t) = e^{at}u(t)$, $a>0$

$$X(j\omega) = \int_0^{\infty} e^{at}\, e^{-j\omega t}\, dt$$

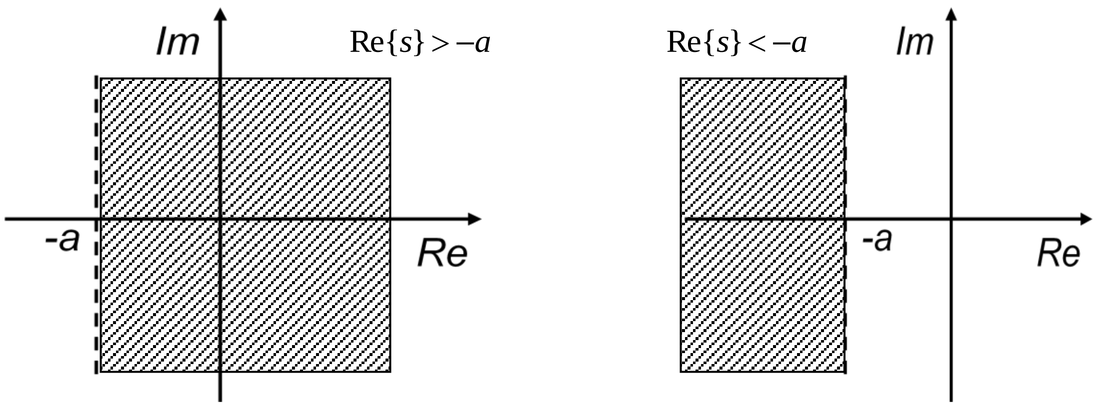
does not exist (is infinite) because the signal's energy is infinite

This is because we multiply $x(t)$ by a complex sinusoidal signal which has unit magnitude for all t and integrate for all time. Therefore, as the *Dirichlet convergence* conditions say, the Fourier transform exists for most signals with finite energy.

Region of Convergence:

The Region Of Convergence (ROC) of the Laplace transform is the set of values for s (=s+$j\omega$) for which the Fourier transform of $x(t)e^{-\sigma t}$ converges (exists).

The ROC is generally displayed by drawing separating line/curve in the complex plane, as illustrated below for Examples 1 and 2, respectively.



The shaded regions denote the ROC for the Laplace transform

Example 4:

Consider a signal that is the sum of two real exponentials:

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

The Laplace transform is then:

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} \left[3e^{-2t}u(t) - 2e^{-t}u(t)\right] e^{-st} dt \\ &= 3\int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st} dt - 2\int_{-\infty}^{\infty} e^{-t}u(t)e^{-st} dt \end{aligned}$$

Using Example 1, each expression can be evaluated as:

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}$$

The ROC associated with these terms are Re{s}>-1 and Re{s}>-2. Therefore, both will converge for Re{s}>-1, and the Laplace transform:

$$X(s) = \frac{s-1}{s^2+3s+2}$$

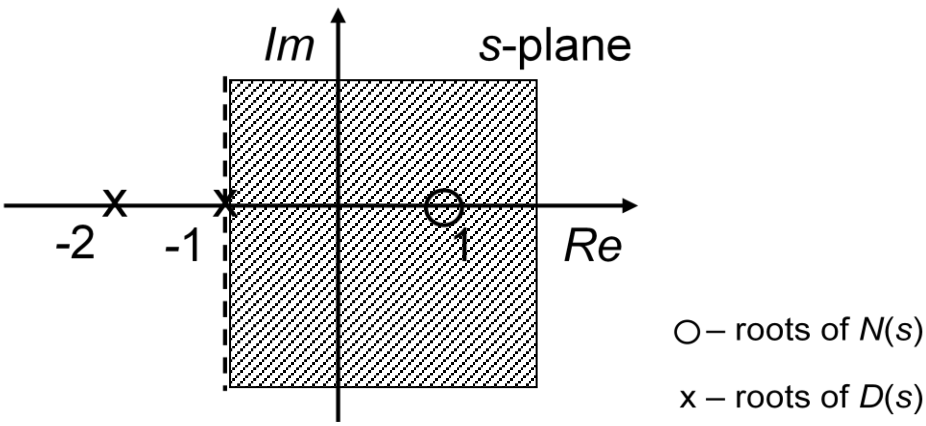
Ratio of Polynomials:
In each of these examples, the Laplace transform is rational, i.e. it is a ratio of polynomials in the complex variable *s*.

$$X(s) = \frac{N(s)}{D(s)}$$

where *N* and *D* are the numerator and denominator polynomial respectively.

In fact, $X(s)$ will be rational whenever $x(t)$ is a linear combination of real or complex exponentials. Rational transforms also arise when we consider LTI systems specified in terms of linear, constant coefficient differential equations.
We can mark the roots of *N* and *D* in the *s*-plane along with the ROC

Example 3:



Poles and Zeros:

The roots of $N(s)$ are known as the ***zeros***. For these values of *s*, $X(s)$ is zero.

The roots of $D(s)$ are known as the ***poles***. For these values of *s*, $X(s)$ is infinite, the Region of Convergence for the Laplace transform cannot contain any poles, because the corresponding integral is infinite.

The set of poles and zeros completely characterise $X(s)$ to within a scale factor (+ ROC for Laplace transform)

$$X(s) \propto \frac{\prod_i (s - z_i)}{\prod_j (s - p_j)}$$

The graphical representation of $X(s)$ through its poles and zeros in the s-plane is referred to as the ***pole-zero*** plot of $X(s)$

Example:
Consider the signal:

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

By linearity we can evaluate the second and third terms

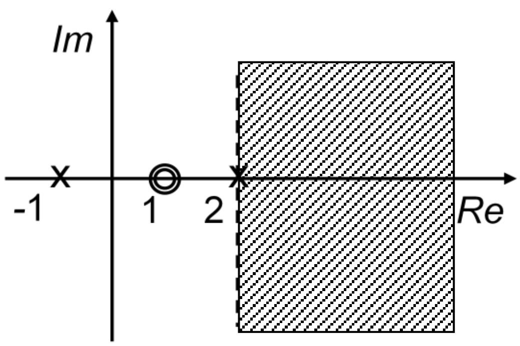
The Laplace transform of the impulse function is:

$$L\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1$$

The Laplace transform of the impulse function is:

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}$$
$$= \frac{(s-1)^2}{(s+1)(s-2)}, \qquad \mathrm{Re}\{s\} > 2$$

which is valid for any *s*. Therefore,



ROC Properties for Laplace Transform:

Property 1: The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the *s*-plane
Because the Laplace transform consists of s for which $x(t)e^{-\sigma t}$ converges, which only depends on $\mathrm{Re}\{s\} = \sigma$
Property 2: For rational Laplace transforms, the ROC does not contain any poles
Because $X(s)$ is infinite at a pole, the integral must not converge.
Property 3: if $x(t)$ is finite duration and is absolutely integrable then the ROC is the entire *s*-plane.
Because $x(t)$ is magnitude bounded, multiplication by any exponential over a finite interval is also bounded. Therefore the Laplace integral converges for any *s*.

Inverse Laplace Transform:

The Laplace transform of a signal $x(t)$ is:

$$X(\sigma + j\omega) = F\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

We can invert this relationship using the inverse Fourier transform

$$x(t)e^{-\sigma t} = F^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

Multiplying both sides by $e^{\sigma t}$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

Therefore, we can recover $x(t)$ from $X(s)$, where the real component is fixed and we integrate over the imaginary part, noting that $ds = jd\omega$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

Inverse Laplace Transform Interpretation:

Just about all real-valued signals, $x(t)$, can be represented as a weighted, $X(s)$, integral of complex exponentials, e^{st}.

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

The contour of integration is a straight line (in the complex plane) from σ-$j\infty$ to σ+$j\infty$ (we won't be explicitly evaluating this, just spotting known transformations). We can choose any s for this integration line, as long as the integral converges

For the class of rational Laplace transforms, we can express $X(s)$ as partial fractions to determine the inverse Fourier transform.

$$X(s) = \sum_{i=1}^{M} \frac{A_i}{s+a_i}$$

$$L^{-1}\{A_i/(s+a_i)\} \to \begin{cases} A_i e^{-a_i t}u(t) & \mathrm{Re}\{s\} > -a_i \\ -A_i e^{-a_i t}u(-t) & \mathrm{Re}\{s\} < -a_i \end{cases}$$

Example 1: Inverting the Laplace Transform

Consider when

$$X(s) = \frac{1}{(s+1)(s+2)} \qquad \Re(s) > -1$$

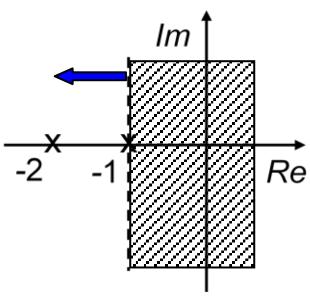
Like the inverse Fourier transform, expand as partial fractions

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

Pole-zero plots and ROC for combined & individual terms

$$e^{-t}u(t) \overset{L}{\leftrightarrow} \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

$$e^{-2t}u(t) \overset{L}{\leftrightarrow} \frac{1}{s+2}, \quad \mathrm{Re}\{s\} > -2$$



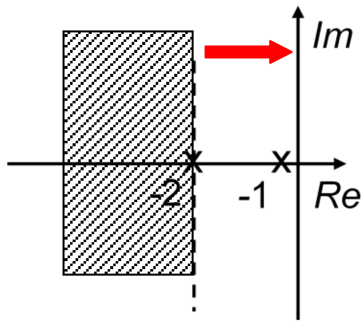
$$x(t) = (e^{-t} - e^{-2t})u(t) \overset{L}{\leftrightarrow} \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$

Example 2: Consider when

$$X(s)=\frac{1}{(s+1)(s+2)}\qquad \mathrm{Re}\{s\}<-2$$

Like the inverse Fourier transform, expand as partial fractions

$$X(s)=\frac{1}{(s+1)(s+2)}=\frac{A}{(s+1)}+\frac{B}{(s+2)}=\frac{1}{(s+1)}-\frac{1}{(s+2)}$$



Pole-zero plots and ROC for combined & individual terms

$$-e^{-t}u(-t)\overset{L}{\leftrightarrow}\frac{1}{s+1},\qquad \mathrm{Re}\{s\}<-1$$

$$-e^{-2t}u(-t)\overset{L}{\leftrightarrow}\frac{1}{s+2},\qquad \mathrm{Re}\{s\}<-2$$

$$x(t)=(-e^{-t}+e^{-2t})u(-t)\overset{L}{\leftrightarrow}\frac{1}{(s+1)(s+2)},\qquad \mathrm{Re}\{s\}<-2$$

Laplace Transform Properties:

a) Linearity property:

If $\quad x_1(t)\overset{L}{\leftrightarrow}X_1(s)\quad$ ROC = R_1

and $\quad x_2(t)\overset{L}{\leftrightarrow}X_2(s)\quad$ ROC = R_2

then $\quad ax_1(t)+bx_2(t)\overset{L}{\leftrightarrow}aX_1(s)+bX_2(s)\quad$ ROC= $R_1\cap R_2$

This follows directly from the definition of the Laplace transform (as the integral operator is linear). It is easily extended to a linear combination of an arbitrary number of signals.

b) Time Shifting property:

If $\quad x(t)\overset{L}{\leftrightarrow}X(s)\quad$ ROC=R

then $\quad x(t-t_0)\overset{L}{\leftrightarrow}e^{-st_0}X(s)\quad$ ROC=R

Proof:

$$x(t)=\tfrac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds$$

Now replacing t by t-t_0

$$x(t-t_0)=\tfrac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{s(t-t_0)}ds$$

$$=\tfrac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}(e^{-st_0}X(s))e^{st}ds$$

Recognising this as $\quad L\{x(t-t_0)\}=e^{-st_0}X(s)$

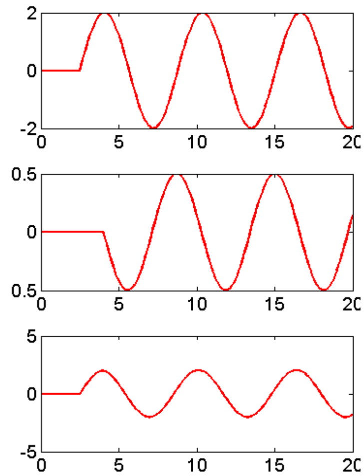
A signal which is shifted in time may have both the magnitude and the phase of the Laplace transform altered.

Example: Linear and Time Shift

Consider the signal (linear sum of two time shifted sinusoids)

$$x(t) = 2x_1(t-2.5) - 0.5x_1(t-4)$$

where $x_1(t) = \sin(\omega_0 t)u(t)$.



Using the sin() Laplace transform example

$$X_1(s) = \frac{\omega_0}{s^2 + \omega_0^2} \qquad \mathrm{Re}\{s\} > 0$$

Then using the *linearity* and *time shift* Laplace transform properties

$$X(s) = \left(2e^{-2.5s} - 0.5e^{-4s}\right)\frac{\omega_0}{s^2 + \omega_0^2} \qquad \mathrm{Re}\{s\} > 0$$

c) Convolution property:

The Laplace transform also has the multiplication property, i.e.

$$x(t) \overset{L}{\leftrightarrow} X(s) \qquad \text{ROC} = R_1$$

$$h(t) \overset{L}{\leftrightarrow} H(s) \qquad \text{ROC} = R_2$$

$$x(t) * h(t) \overset{L}{\leftrightarrow} X(s)H(s) \qquad \text{ROC} \supseteq R_1 \cap R_2$$

roof is "identical" to the Fourier transform convolution

Note that pole-zero cancellation may occur between H(s) and X(s) which extends the ROC

$$X(s) = \frac{s+1}{s+2} \qquad \Re\{s\} > -2$$

$$H(s) = \frac{s+2}{s+1} \qquad \Re\{s\} > -1$$

$$X(s)H(s) = 1 \qquad -\infty < \Re\{s\} < \infty$$

Example 1: First order input & First order system impulse response

Consider the Laplace transform of the output of a first order system when the input is an exponential (decay?)

$$x(t) = e^{-at}u(t)$$

$$h(t) = e^{-bt}u(t)$$

(Solved with Fourier transforms when a,b>0)

Taking Laplace transforms

$$X(s) = \frac{1}{s+a} \qquad \mathrm{Re}\{s\} > -a$$

$$H(s) = \frac{1}{s+b}, \qquad \mathrm{Re}\{s\} > -b$$

Laplace transform of the output is

$$Y(s) = \frac{1}{s+a}\frac{1}{s+b} \qquad \mathrm{Re}\{s\} > \max\{-a,-b\}$$

Splitting into partial fractions

$$Y(s)=\left(\frac{1}{b-a}\right)\left(\frac{1}{s+a}-\frac{1}{s+b}\right) \qquad \mathrm{Re}\{s\}>\max\{-a,-b\}$$

and using the inverse Laplace transform

$$y(t)=\tfrac{1}{b-a}\left(e^{-at}u(t)-e^{-bt}u(t)\right)$$

Note that this is the same as was obtained earlier, expect it is valid for all *a* & *b*, i.e. we can use the Laplace transforms to solve ODEs of LTI systems, using the system's impulse response

$$h(t)\overset{L}{\leftrightarrow}H(s)$$

Example 2: Sinusoidal Input

Consider the 1st order (possible unstable) system response with input $x(t)$

$$h(t)=e^{-at}u(t)$$
$$x(t)=\cos(\omega_0 t)u(t)$$

Taking Laplace transforms

$$H(s)=\frac{1}{s+a} \qquad \mathrm{Re}\{s\}>-a$$
$$X(s)=\frac{s}{s^2+\omega_0^2} \qquad \mathrm{Re}\{s\}>0$$

The Laplace transform of the output of the system is therefore

$$Y(s)=\frac{s}{s^2+\omega_0^2}\frac{1}{s+a} \qquad \mathrm{Re}\{s\}>\max\{0,-a\}$$
$$=\left(\frac{1}{a^2+\omega_0^2}\right)\frac{as+\omega_0^2}{s^2+\omega_0^2}+\left(\frac{-a}{a^2+\omega_0^2}\right)\frac{1}{s+a}$$

and the inverse Laplace transform is

$$y(t)=\frac{u(t)}{a^2+\omega_0^2}\left(a\sin(\omega_0 t)+\omega_0\cos(\omega_0 t)-ae^{-at}\right)$$

***d) Differentiation in the Time Domain*:**

Consider the Laplace transform derivative in the time domain

$$x(t)\overset{L}{\leftrightarrow}X(s) \qquad \text{ROC = R}$$
$$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds$$
$$\frac{dx(t)}{dt}=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}sX(s)e^{st}ds$$
$$\frac{dx(t)}{dt}\overset{L}{\leftrightarrow}sX(s) \qquad \text{ROC}\supseteq\text{R}$$

$sX(s)$ has an extra zero at 0, and may cancel out a corresponding pole of $X(s)$, so ROC may be larger

Widely used to solve when the system is described by LTI differential equations

Example: System Impulse Response

Consider trying to find the system response (potentially unstable) for a second order system with an impulse input $x(t)=\delta(t)$, $y(t)=h(t)$

$$a\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+cy(t)=x(t)$$

Taking Laplace transforms of both sides and using the linearity property

$$aL\left\{\frac{d^2y(t)}{dt^2}\right\}+bL\left\{\frac{dy(t)}{dt}\right\}+cL\{y(t)\}=L\{\delta(t)\}$$

$$L\{y(t)\}(as^2+bs+c)=1$$

$$L\{y(t)\}=H(s)=\frac{1}{as^2+bs+c}=\frac{1}{a(s-r_1)(s-r_2)}=\frac{k_1}{(s-r_1)}+\frac{k_2}{(s-r_2)}$$

where r_1 and r_2 are distinct roots, and calculating the inverse transform

$$y(t)=k_1e^{r_1t}u(t)+k_2e^{r_2t}u(t)$$

The general solution to a second order system can be expressed as the sum of two complex (possibly real) exponentials.

The z-Transform

It plays the same role in the analysis of discrete time signals & LTI systems as the Laplace transform does in the analysis of continuous time signal and LTI systems. The most important one is, in the z-domain, the convolution of two time domain signals is equivalent to multiplication of their corresponding z transforms.

Definition:
The z-transform of a discrete-time signal x[n] is:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

We denote the z-transform operation as

$$x[n] \longleftrightarrow X(z).$$

In general, the number z in *X(z)* is a complex number. Therefore, we may write z as

$$z = re^{jw},$$

where r is the radius of the circle. When r = 1, (7.1) becomes

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

which is the discrete-time Fourier transform of x[n]. Therefore, DTFT is a special case of the z-transform. Hence, we can view DTFT as the z-transform is evaluated on the unit circle. (figure below)

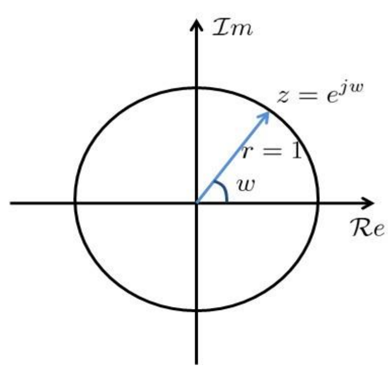


Figure 7.1: Complex z-plane. The z-transform reduces to DTFT for values of z on the unit circle.

When $r \neq 1$, the z-transform is equivalent to

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)(r^{-n}e^{-j\omega n})$$

$$= \sum_{n=-\infty}^{\infty} [x(n)\, r^{-n}]e^{-j\omega n}$$

$$= \mathscr{F}\ [r^{-n}x(n)],$$

which is the DTFT of the signal $r^{-n}x[n]$. However, from the development of DTFT we know that DTFT does not always exist. It exists only when the signal is square summable, or satisfies the Dirichlet conditions. Therefore, $X(z)$ does not always converge. It converges only for some values of r. This range of r is called the region of convergence.

ROC: The Region of Convergence (ROC) of the z-transform is the value of z such that $X(z)$ converges, i.e.,

$$\sum_{n=-\infty}^{\infty} |\ x(n)\ |\, r^{-n} < \infty$$

Example1: Consider the signal $x[n] = a^{n}u[n]$, with $0 < a < 1$. The z-transform of $x[n]$ is

$$X(z) = \sum_{-\infty}^{\infty} a^n\, u[n\,]z^{-n}$$

$$= \sum_{n=0}^{\infty} (\,az^{-1})^n$$

Therefore, X(z) converges if $\sum_{n=0}^{\infty}(a\, z^{-1})^n < \infty$. From geometric series, we know that

$$\sum_{n=0}^{\infty}(r\, z^{-1})^n = \frac{1}{1-az-1}$$

when $|az^{-1}| < 1$, or equivalently $|z| > |a|$. So,

$$X(z) = \frac{1}{1-ax^{-1}},$$

with ROC being the set of z such that $|z| > |a|$.

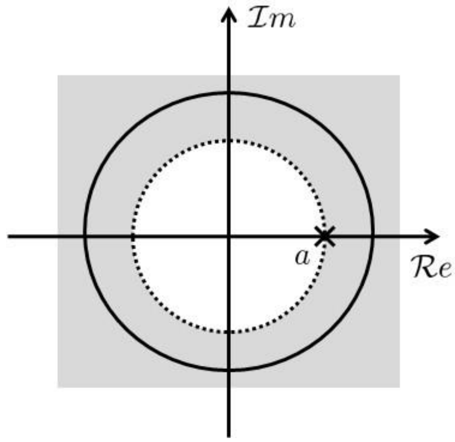


Figure 7.2: Pole-zero plot and ROC of Example 1.

Note: ROC of casual and infinite sequence is the exterior of a circle having radius |a|.

Example2: Consider the signal $x[n] = -a^n u[-n-1]$ with $0 < a < 1$. The z-transform of x[n] is

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n\, u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n\, z^{-n}$$

$$= -\sum_{n=-1}^{\infty} a^{-n}\, z^{n}$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}\, z)^n$$

Therefore, X(z) converges when $|a^{-1}z| < 1$, or equivalently $|z| < |a|$. In this case,

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}},$$

with ROC being the set of z such that $|z| < |a|$. Here, the z-transform is same as that of Example 1, the only difference being ROC. Example 2 is just the left-sided version of Example 1.

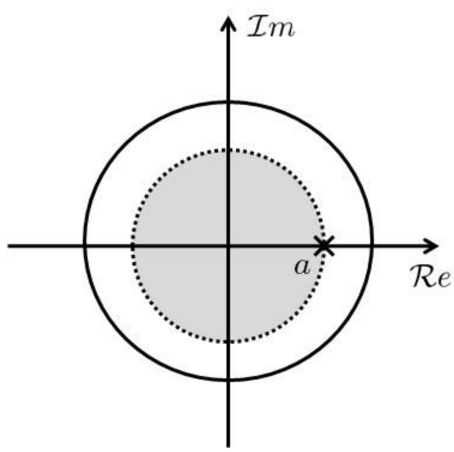


Figure 7.3: Pole-zero plot and ROC of Example 2.

Properties of ROC

(1) The ROC is a ring or disk in the z-plane centered at origin.

(2) DTFT of x[n] exists if and only if ROC includes the unit circle.

Proof: By definition, ROC is the set of z such that X (z) converges. DTFT is the z-transform evaluated on the unit circle. Therefore, if ROC includes the unit circle, then X (z) converges for any value of z on the unit circle i.e., DTFT converges.

(3) The ROC does not contain any pole.
(4) If x[n] is a right-sided sequence, then ROC extends outward from the outermost pole.
(5) If x[n] is a left-sided sequence, then ROC extends inward from the innermost pole.

(6) If x (n) is a causal sequence, then the ROC is the entire z plane except at $z = 0$.

(7) If x (n) is a non-causal sequence, then the ROC is the entire z plane except at $z = \infty$.

(8) If x (n) is an infinite duration, two sided sequence; the ROC will consist of a ring in the z plane, bounded on the interior or exterior by a pole, not containing any pole.
(9) The ROC of a LTI stable system contains the unit circle.

Properties of z-transform

- Linearity:
  If $x_1(n) \leftrightarrow X_1(z)$ and $x_2(n) \leftrightarrow X_2(z)$, then
  $ax_1[n] + bx_2[n] \longleftrightarrow aX_1(z) + bX_2(z)$

- Time reversal:
  If $x(n) \leftrightarrow X(z)$, then
  $x(-n)\ ) \leftrightarrow X(z^{-1})$

- Time shift:
  If $x(n) \leftrightarrow X(z)$, then
  $x[n - n_0\ ] \longleftrightarrow X\ (z)z^{-n_0}$, Where n_0 is an integer.

- Multiplication by α^n:
  If $x(n) \leftrightarrow X(z)$, then
  $[a^n x(n)]\ x(n) \leftrightarrow X(z)\ X(a^{-1}z)$

- Convolution:
  If $x(n) \leftrightarrow X(z)$, $h(n) \leftrightarrow H(z)$, then
  $[x(n) * h(n)] \leftrightarrow X(z)\ H(z)$

- Differentiation in the z-domain:
  If $x(n) \leftrightarrow X(z)$, then
  $[nx(n)]\ ) \leftrightarrow -z \frac{d}{dz} X(z)$

z-transform Pairs

7.2.1 A. Table 10.2

1. $\delta[n] \longleftrightarrow 1$, all z
2. $\delta[n-m] \longleftrightarrow z^{-m}$, all z except 0 when $m > 0$ and ∞ when $m < 0$.
3. $u[n] \longleftrightarrow \frac{1}{1-z^{-1}}$, $|z| > 1$
4. $a^n u[n] \longleftrightarrow \frac{1}{1-az^{-1}}$, $|z| > a$
5. $-a^n u[-n-1] \longleftrightarrow \frac{1}{1-az^{-1}}$, $|z| < |a|$.
6. $x^*[n] \longleftrightarrow X^*(z^*)$

Inverse z-transform:

Three different methods are:
(1) Partial fraction method
(2) Power series method
(3) Long division method

Partial fraction method:

- In case of LTI systems, commonly encountered form of z-transform is

$$X(z) = \frac{B(z)}{A(z)}$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

Usually M<N

- If M>N, then long division method is used & X(z) is expressed in the form

$$X(z) = \sum_{k=0}^{M-N} f_k z^{-k} + \frac{B(z)}{A(z)}$$

Where B(z) has now the order one less than the denominator polynomial and partial fraction method is used to find z transform.

- The inverse z-transform of the terms in the summation are obtained from the transform pair and time shift property

$$\delta[n] \longleftrightarrow 1$$

$$\delta[n-m] \longleftrightarrow z^{-m}$$

- If X(z) is expressed as ratio of polynomials in z instead of z^{-1}, then it is converted to the polynomial of z^{-1}.
- Convert the denominator into the product of first-order terms

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Where d_k are the poles of X(z).

For distinct poles:

- For all distinct poles, the X(z) can be written as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{(1 - d_k z^{-1})}$$

- Depending on ROC, the inverse z-transform associated with each term is then is determined by using the appropriate transform pair. We get

$$A_k(d_k)^n u[n] \longleftrightarrow \frac{A_k}{(1-d_k z^{-1})} \text{ With ROC z > dk}$$

$$-A_k(d_k)^n u[-n-1] \longleftrightarrow \frac{A_k}{(1-d_k z^{-1})} \text{ With ROC z < dk}$$

- For each term the relationship between the ROC associated with X(z) and each pole determines whether the right sided or left sided inverse transform is selected.

For repeated poles:

For Repeated poles

- If pole d_i is repeated r times, then there are r terms in the partial fraction expansion associated with that pole.

$$\frac{Ai_1}{1 - d_i z^{-1}}, \frac{Ai_1}{(1 - d_i z^{-1})^2} \cdots \cdots \frac{Ai_1}{(1 - d_i z^{-1})^r}$$

- Here if X(z) with ROC $|z| > d_i$, then the right sided inverse transform is chosen & if X(z) with ROC $|z| < d_i$, then the left sided inverse transform is chosen.

Example: Find the inverse z transform of

$$X(z) = \frac{1 + 3z^{-1}}{1 + 3z^{-1} + a_2 z^{-2}} \quad |z| > 2$$

Solution: First we eliminate the negative power by multiplying numerator & denominator by z^2.

$$X(z) = \frac{z(z+3)}{z^2+3z+2}$$

Dividing X(z) by z we get

$$\frac{X(z)}{z} = \frac{z+3}{(z+1)(z+2)}$$

The above equation can be written in partial fraction form

$$\frac{X\,(z)}{z} = \frac{C_1}{(z+1)} + \frac{C_2}{(z+2)}$$

Where, $C_1 = (z+1)\frac{X(z)}{z}\,|_{z=-1} = 2$

Similarly, $C_1 = (z+2)\frac{X(z)}{z}\,|_{z=-2} = -1$

Therefore, $\frac{X\,(z)}{z} = \frac{2}{(z+1)} - \frac{1}{(z+2)}$

$$X(z) = 2\frac{z}{(z+1)} - \frac{z}{(z+2)}$$

As ROC is $|z| > 2$, the sequence is causal and we find

$$x(n) = 2\,(-1)^n u(n) - (-2)^n u(n)$$

Power series method:

- Express X(z) as a power series in z^{-1} or z as given in z-transform equation.
- The values of the signal x[n] are then given by coefficient associated with z^{-n}.
- Main disadvantage is : limited to one sided signals.
- Signals with ROCs of the form $|z| > a$ or $|z| < a$.
- If ROC is $|z| > a$, then express X(z) as a power series in z^{-1} and we get right sided signal.
- If ROC is $|z| < a$, then express X(z) as a power series in z and we get left sided signal.

Long division method:

- Find the z-transform of

$$X(z) = \frac{2+z^{-1}}{1-\frac{1}{2}z^{-1}}\text{ , with ROC } |z| > \frac{1}{2}$$

- Solution: since ROC indicates that x[n] is right sided sequence, use long division method to write X(z) as a power series in z^{-1}.

- We get

$$X(z) = 2+2z^{-1}+z^{-2}+\frac{1}{2}z^{-3}+\ldots\ldots$$

- Comparing with z-transform pair, we get

$$x(n) = 2\delta(n) + 2\delta(n-1) + \delta(n-2) + \frac{1}{2}\delta(n-3) + \ldots\ldots\ldots$$

- If we change the ROC to $|z| < \frac{1}{2}$, then expand X(z) as a power series in z using long division method , we get

$$X(z) = -2 - 8z - 16z^2 - 32z^{-3} + \ldots\ldots$$

- Now we get x(n) as

$$x(n)= -2\delta(n)-8\delta(n+1)-16\delta(n+2)+ 32\delta(n+3)+\ldots\ldots\ldots$$