MATHEMATICS-II

UNIT-I

For the sake of simplicity, all the theories are outlined for a real valued function of three variables which are also true for a real valued function of two variables.

1 Vector algebra in 2-space and 3-space

A **vector** is a quantity that is determined by both its magnitude and its direction. A vector is usually given by a initial point and a terminal point. If a given vector v has initial point $P : (x_1, y_1, z_1)$ and terminal point $Q = (x_2, y_2, z_2)$, the three numbers $v_1 = x_2 - x_1, v_2 = y_2 - y_1$, and $v_3 = z_2 - z_1$ are called components of v and we write simply

$$v = (v_1, v_2, v_3).$$

Length of v is

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Two vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are added as follows:

$$v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3).$$

A scalar c is multiplied to the vector $v = (v_1, v_2, v_3)$ as follows:

$$cv = (cv_1, cv_2, cv_3).$$

Basic properties of vector addition and scalar multiplication

(a) $v + w = w + v$

(b) $(v + w) + z = v + (w + z)$

(c) $v + 0 = 0 + v = v$

(d) $v + (-v) = 0$

(e) $c(v + w) = cv + cw$

(f) $(c + d)v = cv + dv$

(g) $c(dv) = (cd)v$

(h) $1v = v$

(i) $0v = 0$

(j) $(-1)v = -v$

A vector has length 1 is called a unit vector. The unit vectors along the direction X-axis, Y-axis and Z-axis are i, j and k, respectively. In component form $i = (1, 0, 0)$, $j = (0, 1, 0)$ and $k = (0, 0, 1)$. Another popular representation of a vector $v = (v_1, v_2, v_3)$ is $v = v_1 i + v_2 j + v_3 k$.

Inner products of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is defined by

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

General properties of inner products

(a) $(au + bv) \cdot w = au \cdot w + bv \cdot w$ (Linearity)

(b) $u \cdot v = v \cdot w$ (Commutativity)

(c) $u \cdot u \geq 0$. $u \cdot u = 0$ if and only if $u = 0$. (Positivity)

(d) $(u + v) \cdot w = u \cdot w + v \cdot w$ (Distributive)

(e) $|u \cdot v| \leq |u|\,|v|$ (Cauchy-Schwarz inequality)

(f) $|u + v| \leq |u| + |v|$ (Triangle inequality)

Two vectors u and v are orthogonal if $u \cdot v = 0$.

Some special examples of inner products are

$$i \cdot i = 1, j \cdot j = 1, k \cdot k = 1,$$

and

$$i \cdot j = 0, j \cdot k = 0, k \cdot i = 0.$$

Let θ is the angle between u and v, then

$$cos(\theta) = \frac{u \cdot v}{|u|\,|v|}.$$

Applications of inner products

Example 1.1. Work done by a force p on a body giving displacement d is $p \cdot d$.

Example 1.2. Component of a force p in a given direction d is $\frac{p \cdot d}{|p|}$.

Example 1.3 (Orthogonal straight lines in the plane)**.** Find the straight line L_1 through the point $P(1, 3)$ in the xy-plane and perpendicular to the straight line $x - 2y + 2 = 0$.

Let L_1 be the straight line $ax + by = c$, and L_1^* be the straight line $ax + by = 0$. L_1^* passes through the origin and is parallel to L_1. Let us denote the straight line $x - 2y + 2 = 0$ by L_2 and $x - 2y = 0$ by L_2^*. L_2^* passes through the origin and is parallel to L_2. The point $(b, -a)$ lies on L_1^*. Similarly, the point $(2, 1)$ lies on L_2^*. Therefore, L_2^* and L_1^* are perpendicular to each other if $(2, 1) \cdot (b, -a) = 0$, for instance, if$a = 2, b = 1$. Therefore, L_1 is the straight line $2x + y = c$. Since, it passes through $(1, 3)$ $c = 5$. Hence the desired straight line is $2x + y = 5$.

Example 1.4 (Normal vector to a plane)**.** Find a unit vector perpendicular to the plane $4x + 2y + 4z = -7$.

We may write any plane in space as $a \cdot r = a_1x + a_2y + a_3z = c$, where $a = (a_1, a_2, a_3) \neq 0$ and $r = (x, y, z)$. Normalizing, we get $n \cdot r = p$, where $p = \frac{c}{|a|}$ and $n = \frac{a}{|a|}$. This n is the unit vector normal to the given plane. In our example, $n = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.

Vector product (Cross product)

The vector product or cross product of two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ is the vector $v = (v_1, v_2, v_3) = a \times \times b$, where

$$v_1 = a_2b_3 - a_3b_2, v_2 = a_3b_1 - a_1b_3, v_3 = a_1b_2 - a_2b_1.$$

It is easy to remember the vector product $a \times b$ by the symbolical determinant formula

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Vector product of standard basis vectors

$$i \times j = k, j \times k = i, k \times i = j,$$
$$j \times i = -k, k \times j = -i, i \times k = -j.$$

General properties of vector product

1. $(la) \times b = l(a \times b) = a \times (lb)$
2. $a \times (b + c) = a \times b + a \times c$ (distributive)
3. $(a + b) \times c = a \times c + b \times c$ (distributive)
4. $a \times b = -b \times a$ (anticommutative)
5. $a \times (b \times c) \neq (a \times b) \times c$ (not associative)

Applications of vector product

Example 1.5 (Moment of a force)**.** Let a force p acts on a line through a point A. The moment vector about a pint Q is

$$m = r \times p,$$

where r is the vector whose initial point is Q and the terminal point is A.

Example 1.6. Velocity of a rotating body B rotating with angular velocity w is

$$v = w \times r,$$

where r is the position vector of any point on B referred to a coordinate system with origin 0 on the axis of rotation.

Scalar triple product

The scalar triple product of three vectors $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$ is denoted by (a, b, c) and is defined by

$$(a, b, c) = a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

An important property of scalar triple product is $a \cdot (b \times c) = (a \times b) \cdot c$.

Theorem 1.7. *Three vectors form a linearly independent set if and only if their scalar triple product is not zero.*

Here are some important facts:

- The modulus of $a \times b$ is the area of a parallelogram with a and b as the adjacent sides.
- The absolute value of the scalar triple product $|(a, b, c)|$ is the volume of the parallelepiped with a, b and c as the concurrent edges.
- The volume of the tetrahedron is 1/6 of the volume of the parallelepiped.
- The area of a triangle is 1/2 of the area of the parallelepiped.

PROBLEMS

1. Find the area of the parallelogram if the vertices are $(1, 1), (4, -2), (9, 3), (12, 0)$.
2. Find the area of the triangle in space if the vertices are $(1, 3, 2), (3, -4, 2)$ and $(5, 0, -5)$.

2 Vector differential calculus, basic definitions

Definition 2.1 (Gradient)**.** The gradient of a scalar field ϕ is the vector field $\nabla\phi$ given by

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k,$$

whenever these partial derivatives are defined.

Example 2.2. Let $\phi(x,y,z) = x^2y\cos(yz)$. Then

$$\nabla\phi = 2xy\cos(yz)i + \left(x^2\cos(yz) - x^2yz\sin(yz)\right)j - x^2y^2\sin(yz)k.$$

The gradient field evaluated at a point P is denoted by $\nabla\phi(P)$. For the gradient just computed,

$$\nabla\phi(1,-1,3) = -2\cos(3)i + [\cos(3) - 3\sin(3)]j + \sin(3)k.$$

Definition 2.3 (Divergence)**.** The divergence of a vector field $F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ is the scalar field

$$divF = \nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

For example, if $F = 2xyi + (xyz^2 - \sin(yz))j + ze^{x+y}k$, then

$$divF = 2y + xz^2 - z\cos(yz) + e^{x+y}.$$

Definition 2.4 (Curl)**.** The curl of a vector field $F(x,y,z) = f(x,y,z)i+g(x,y,z)j+h(x,y,z)k$ is the vetor filed

$$curlF = \nabla \times F = (\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z})i + (\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x})j + (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})k.$$

Foe example, if $F = yi + 2xzj + ze^xk$, then

$$curlF = -2xi - ze^zj + (2z-1)k.$$

Theorem 2.5. *Let ϕ be continuous in its first and second partial derivatives then $curl(grad\phi) = 0$.*

Proof. By direct computation

$$\begin{aligned}
\nabla \times \nabla\phi &= \nabla \times (\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k) \\
&= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} \\
&= 0.
\end{aligned}$$

□

Theorem 2.6. *Let F be a continuous vector field whose components have continuous first and second partial derivatives. Then*

$$div(curl F) = 0.$$

Proof. By direct computation

$$\begin{aligned}\nabla \cdot (\nabla \times F) &= \frac{\partial}{\partial x}(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) \\ &= 0.\end{aligned}$$

□

3 Derivatives, directional derivatives

Definition 3.1 (Directional derivative)**.** The directional derivative of a scalar field ϕ at P_0 in the direction of the unit vector $u = ai + bj + ck$ is denoted by $D_u\phi(P_0) = \frac{d}{dt}\phi(x + at, y + bt + z + ct)\big|_{t=0}$.

We usually compute a directional derivative using the following theorem.

Theorem 3.2. *If ϕ is a differentiable function of three variables, and u is a constant unit vector, then*

$$D_u\phi(P_0) = \nabla\phi(P_0) \cdot u.$$

Example 3.3. Let $\phi(x, y, z) = x^2y - xe^z$, $P_0 = (2, -1, \pi)$ and $u = \frac{1}{\sqrt{6}}(i - 2j + k)$. Then the rate of change of $\phi(x, y, z)$ at P_0 in the direction of u is the directional derivative

$$\begin{aligned}D_u\phi(2, -1, \pi) &= \nabla\phi(2, -1, \pi) \cdot u \\ &= \phi_x(2, -1, \pi)\frac{1}{\sqrt{6}} + \phi_y(2, -1, \pi)\frac{-2}{\sqrt{6}} + \phi_z(2, -1, \pi)\frac{1}{\sqrt{6}} \\ &= \frac{1}{\sqrt{6}}\left([2xy - e^z]_{(2,-1,\pi)} - 2[x^2]_{(2,-1,\pi)} + [-xe^z]_{(2,-1,\pi)}\right) \\ &= \frac{-3}{\sqrt{6}}(4 + e^\pi).\end{aligned}$$

We will now show that the gradient vector $\nabla\phi(P_0)$ points in the direction of maximum rate of increase at P_0, and $-\nabla\phi(P_0)$ in the direction of minimum rate of increase.

Theorem 3.4. *Let ϕ and its first partial derivatives be continuous in some sphere about P_0, and suppose that $\nabla\phi(P_0) \neq 0$. Then*

1. *At P_0, $\phi(x, y, z)$ has its maximum rate of change in the direction of $\nabla\phi(P_0)$. This maximum rate of change is $\|\nabla\phi(P_0)\|$.*

2. *At P_0, $\phi(x, y, z)$ has its minimum rate of change in the direction of $-\nabla\phi(P_0)$. This minimum rate of change is $-\|\nabla\phi(P_0)\|$.*

Example 3.5. Let $\phi(x, y, z) = 2xz+e^y z^2$. We will find the maximum and minimum rates of change of $\phi(x, y, z)$ from $(2, 1, 1)$. First,

$$\nabla\phi(x, y, z) = 2zi + e^y z^2 j + (2x + 2ze^y)k,$$

so

$$\nabla\phi(P_0) = 2i + ej + (4 + 2e)k.$$

The maximum rate of increase of $\phi(x, y, z)$ at $(2, 1, 1)$ is in the direction of this gradient, and this maximum rate of change is $\sqrt{4 + e^2 + (4 + 2e)^2}$.

The minimum rate of increase of $\phi(x, y, z)$ at $(2, 1, 1)$ is in the direction of $-2i - ej - (4 + 2e)k$, and this minimum rate of change is $-\sqrt{4 + e^2 + (4 + 2e)^2}$.

PROBLEMS

Compute the gradient of the function and evaluate this gradient at the given point. Determine at this point the maximum and minimum rate of change of the function.

1. $\phi(x, y, z) = xyz$; (1,1,1)

2. $\phi(x, y, z) = x^2y - sin(xz)$; $(1, -1, \pi/4)$

3. $\phi(x, y, z) = 2xy + xe^z$; (-2,1,6)

4. $\phi(x, y, z) = cos(xyz)$; $(-1, 1, \pi/2)$

4 Gradient of a scalar field

A real-valued function of three variables is called a scalar field. Depending on the function ϕ and the constant k, the locus of points $\phi(x, y, z) = k$ may form a surface in 3-space. Any such surface is called a **level surface** of ϕ. For example, if

$\phi(x, y, z) = x^2 + y^2 + z^2$ and $k > 0$, then the level surface $\phi(x, y, z) = k$ is a sphere of radius $\sqrt{k}$. If $k = 0$ this locus is just a single point, the origin. If $k < 0$ this locus is empty.

Theorem 4.1. *Let ϕ and its partial derivative be continuous. Then $\nabla\phi(P)$ is normal to the level surface $\phi(x, y, z) = k$ at any point P on this surface at which this gradient vector is nonzero.*

Once we have this normal vector, the **equation of the tangent plane** passing through $P_0 = (x_0, y_0, z_0)$ is obtained as

$$\nabla\phi(P_0) \cdot [(x - x_0)i + (y - y_0)j + (z - z_0)k] = 0.$$

Simplifying, we get

$$\frac{\partial\phi}{\partial x}(P_0)(x - x_0) + \frac{\partial\phi}{\partial y}(P_0)(y - y_0) + \frac{\partial\phi}{\partial z}(P_0)(z - z_0) = 0. \tag{1}$$

The parametric equation of the normal line at P_0 is

$$x - x_0 = \frac{\partial\phi}{\partial x}(P_0)t, y - y_0 = \frac{\partial\phi}{\partial y}(P_0)t, z - z_0 = \frac{\partial\phi}{\partial z}(P_0)t,$$

where t varies over the real line.

Here is an example.

Example 4.2. Consider the (level) surface $\phi(x, y, z) = z - \sqrt{x^2 + y^2} = 0$. This surface is a cone with vertex at $(0, 0, 0)$. The gradient vector is

$$\nabla\phi = -\frac{x}{\sqrt{x^2 + y^2}}i - \frac{y}{\sqrt{x^2 + y^2}}j + k,$$

provided both x and y are nonzero. the point $(1, 1, \sqrt{2})$ is a point on the cone. At this point the gradient vector is $\nabla\phi(1, 1, \sqrt{2}) = -\frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}j + k$. This is the normal vector to the cone at $(1, 1, \sqrt{2})$. The tangent plane at this point has equation

$$-\frac{1}{\sqrt{2}}(x - 1) - \frac{1}{\sqrt{2}}(y - 1) + z - \sqrt{2} = 0,$$

or

$$x + y - \sqrt{2}z = 0.$$

The normal line through this point has parametric equation

$$x - 1 = -\frac{1}{\sqrt{2}}t, y - 1 = -\frac{1}{\sqrt{2}}t, z - \sqrt{2} = t.$$

Simplifying, we get

$$x = 1 - \frac{1}{\sqrt{2}}t, y = 1 - \frac{1}{\sqrt{2}}t, z = \sqrt{2} + t.$$

PROBLEMS

Find the equations of the tangent plane and normal line to the surface at the given point.

1. $x^2 + y^2 + z^2 = 4$; $(1, 1, \sqrt{2})$
2. $z = x^2 - y^2$; (1,1,0)
3. $x^2 - y^2 + z^2 = 0$; (1,1,0)
4. $3x^4 + 3y^4 + 6z^4 = 12$; (1,1,1)

5 Physical interpretation of divergence and curl of a vector field

Physical interpretation of divergence

Suppose $F(x, y, z, t)$ is the velocity of a fluid at point (x, y, z) and time t. Time plays no role in computing divergence, but it is include here because, normally a velocity vector does depend on time t. The divergence of $F(x, y, z, t)$ at time t is interpreted as a measure of the outward flow or expansion of the fluid from this point.

In case *div* of a vector field is zero, we call it *incompressible*.

Physical interpretation of curl

The angular velocity of a uniformly rotating body is constant times the curl of the tangential linear velocity. In other words

$$\Omega = \frac{1}{2}\nabla \times T,$$

where Ω is the angular velocity and T is the tangential linear velocity.

In case *curl* of a vector field is zero, we call it *irrotational*.

PROBLEMS

1. Let F and G be vector fields. Prove that

$$\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$$

2. Let ϕ and ψ be scalar fields. Prove that $\nabla \cdot (\nabla\phi \times \nabla\psi) = 0$.

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1. Vector Integral Calculus

Vector integral calculus can be seen as a generalization of regular integral calculus. extends integrals as known from regular calculus to integrals over curves, called line integrals , surfaces, called surface integrals , and solids, called triple integrals Vector integral calculus is very important to the engineer and physicist and has many applications in solid mechanics, in fluid flow, in heat problems, and others.

Line Integrals

The concept of a line integral is a simple and natural generalization of a definite integral

$$\int_a^b f(x)dx \tag{1.1}$$

Recall that, in (1.1), we integrate the function $f(x)$, also known as the integrand, from $x = a$ along the x-axis to $x = b$. Now, in a line integral, we shall integrate a given function, also called the integrand, along a curve C in space or in the plane. (Hence curve integral would be a better name but line integral is standard). This requires that we represent the curve C by a parametric representation

$$r(t) = [x(t), y(t), z(t)] = x(t)i + y(t)j + z(t)k \ \ (a \leq t \leq b). \tag{1.2}$$

The curve C is called the path of integration. The path of integration goes from A to B. Thus $A : r(a)$ is its initial point and $B : r(b)$ is its terminal point. C is now oriented. The direction from $A$to B, in which t increases is called the positive direction on C. We mark it by an arrow. The points A and B may coincide. Then C is called a closed path. C is called a smooth curve if it has at each point a unique tangent whose direction varies continuously as we move along C. We note that $r(t)$ in (1.2) is differentiable. Its derivative $r'(t) = \frac{dr}{dt}$ is continuous and different from the zero vector at every point of C.

Definition and Evaluation of Line Integrals

A line integral of a vector function $F(r)$over a curve $C : r(t)$ is defined by

$$\int_C F(r)dr = \int_a^b F(r(t)).r'(t)dt \tag{1.3}$$

If we write $dr = [dx, dy, dz]$ then (1.3) becomes

$$\int_C F(r(t)).r'(t)dt = \int_C (F_1dx + F_2dy + F_3dz) = \int_a^b (F_1x' + F_2y' + F_3z')dt. \tag{1.4}$$

Example 1.1. *Find the value of the line integral , when$F(r) = [-y, -xy]$ and C is the circular arc of unit cir cle in first quardant.*

solution

We may represent C by $r(t) = [cost, sint]$ *, where* $0 \le t \le \frac{\pi}{2}$*. Then*

$$F(r(t)) = -sinti - costsintj$$

.

$$\int_C F(r(t)).r'(t)dt = \int_0^{\frac{\pi}{2}} (sin^2t - cos^2tsint)dt = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-cos2t)dt - \int_1^0 u^2(-du) = \frac{\pi}{4} - \frac{1}{3}$$

Simple general properties of the line integral

$$\int_C kFdr = kFdr \ (kconstant) \tag{1.5}$$

$$\int_C (F+G)dr = \int_C Fdr + \int_C Gdr \tag{1.6}$$

$$\int_C Fdr = \int_{C1} Fdr + \int_{C2} Fdr \tag{1.7}$$

• Direction-Preserving Parametric Transformations Any representations of C that give the same positive direction on C also yield the same value of the line integral(1.3).

Other Forms of Line Integrals

The line integrals

$\int_C F1dx, \int_C F2dy, \int_C F3dz$

are special cases of(1.3) when $F = F_1iorF_2jorF_3k$, respectively. Furthermore, without taking a dot product as in(1.3) we can obtain a line integral whose value is a vector rather than a scalar, namely,

$$\int_C F(r)dt = \int_a^b F(r(t))dt = \int_a^b [F_1(r(t)), F_2(r(t)), F_3(r(t))]dt. \tag{1.8}$$

Example 1.2. *Integrate*$F(r) = [xy, yz, z]$ *along the helix* $r(t) = [cost, sint, 3t]$ $(0 \le t \le 2\pi)$.

solution

$F(r(t)) = [costsint, 3tsint, 3t]$ *integrated with respect to t from* 0 *to* 2π *gives*

$$\int_0^{2\pi} F(r(t)dt = \left[-\frac{1}{2}cos^2t, 3sint - 3tcost, \frac{3}{2}t^2\right]_0^{2\pi} = [0, -6\pi, 6\pi^2].$$

• **Path Dependence**

Path dependence of line integrals is practically and theoretically so important that we formulate it as a theorem. And this section will be devoted to conditions under which path dependence does not occur. The line integral (1.3) generally depends not only on F and on the endpoints A and B of the path, but also on the path itself along which the integral is taken.

Example 1.3. *Show that the differential form under the integral sign of*

$$I = \int_C \left[2xyz^2 + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz\right]$$

is exact, so that we have independence of path in any domain, and find the value of I *from* $A : (0,0,1) to B : (1, \frac{\pi}{4}, 2)$.

Solution

It is exact as

$$(F_3)_y = 2x^2z + \cos yz - yz\sin yz = (F_2)_z$$
$$(F_1)_z = 4xyz = (F_3)_x$$
$$(F_2)_x = 2xz^2 = (F_1)_y$$
$$f = \int F_2 dy = \int (x^2z^2 + z\cos yz)dy = x^2z^2y + \sin yz + g(x,z)$$
$$f_x = 2xz^2y + g_x = F_1 = 2xyz^2, \quad g_x = 0, \quad g = h(z)$$
$$f_z = 2x^2zy + y\cos yz + h' = F_3 = 2x^2zy + y\cos yz. \quad h' = 0$$

Taking $h = 0$ we get $x^2yz^2 + \sin yz$

And value of the integral

$$I = f(1, \frac{\pi}{4}, 2) - f(0,0,1) = \pi - \sin\frac{\pi}{2} - 0 = \pi + 1$$

Double Integral

Double integrals over a plane region may be transformed into line integrals over the boundary of the region and conversely. This is of practical interest because it may simplify the evaluation of an integral. It also helps in theoretical work when we want to switch from one kind of integral to the other. The transformation can be done by the following theorem.

Example 1.4. *Evalute*

$$\int_0^2 \int_0^4 (x^2 + y^2)dxdy$$

Solution

$$\int_0^2 \int_0^4 (x^2 + y^2)dxdy = \int_0^2 \left[(\frac{x^3}{3} + y^2x)\right]_0^4$$
$$= \int_0^2 (\frac{64}{3} + 4y^2)$$
$$= \left[\frac{64}{3}y + \frac{4y^3}{3}\right]_0^2$$

$$= 64 + 36 = 100$$

Green's Theorem in Plane

Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves . Let $F_1(x,y)$ *and* $F_2(x,y)$ **be functions that are continuous and have continuous partial derivatives** $\frac{\partial F_1}{\partial y}$ **and** $\frac{\partial F_2}{\partial x}$ **everywhere in some domain containing** R**. Then**

$$\int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C (F_1 dx + F_2 dy) \tag{1.9}$$

Here we integrate along the entire boundary C **of** R **in such a sense that** R **is on the left as we advance in the direction of integration.**

Verification of Green's Theorem in the Plane

Let $F_1 = y^2 - 7y$ *and* $F_2 = 2xy + 2x$ **and C the circle** $x^2 + y^2 = 1$**.**

Solution

In (1.9) **on the left we get**

$$\int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \int\int_R [(2y+2) - (2y-7)]\, dxdy = 9\int\int_R dxdy = 9\pi$$

Since the circular disk R **has area** π

We now show that the line integral in (1.9) **on the right gives the same value,** 9π**. We must orient** C **counterclockwise, say,**$r(t) = [cost, sint]$**. Then**

$$r'(t) = [-sint, cost]$$

, and on$C$**,**

$$F_1 = y^2 - 7y = sin^2t - 7sint, \text{ and } F_2 = 2xy + 2x = 2costsint + 2cost.$$

.

Hence the result of line integration

$$\oint (F_1x' + F_2y')dt = \int_0^{2\pi} [(sin^2t - 7sint)(-sint) + 2(costsint + cost)(cost)]dt$$

$$= \int_0^{2\pi} (-sin3t + 7sin^2t + 2cos^2tsint + 2cos^2t)dt$$

$$= 0 + 7\pi - 0 + 2\pi = 9\pi$$

Surfaces for Surface Integrals

With line integrals, we integrate over curves in space , with surface integrals we integrate over surfaces in space. Each curve in space is represented by a parametric equation . This suggests that we should also find

parametric representations for the surfaces in space. This is indeed one of the goals of this section. The surfaces considered are cylinders, spheres, cones, and others. The second goal is to learn about surface normals. Both goals prepare us on surface integrals. Note that for simplicity, we shall say "surface" also for a portion of a surface.

Representation of Surfaces

Representations of a surface S in xyz-space are

$$z = f(x, y) or g(x, y, z) = 0. \tag{1.10}$$

For example, $z = +\sqrt{a^2 - x^2 - y^2}$or $x^2 + y^2 + z^2 - a^2 = 0 \ \ (z \geq 0)$ represents a hemisphere of radius $a$and center 0.

Surface Integrals

To define a surface integral, we take a surface S, given by a parametric representation as just discussed,

$$r(u, v) = [(u, v), y(u, v), z(u, v)] = x(u, v)i + y(u, v)j + z(u, v)k \tag{1.11}$$

where (u, v) varies over a region R in the uv-plane. We assume S to be piecewise smooth , so that S has a normal vector

$$N = r_u \times r_v and unit normal vector n = \frac{\mathbf{N}}{|N|} \tag{1.12}$$

at every point (except perhaps for some edges or cusps, as for a cube or cone). For a given vector function F we can now define the surface integral over S by

$$\int\int_S F n dA = \int\int_R F(r(u, v))N(u, v) du dv \tag{1.13}$$

Here $N = |N|n$ by (1.12), and $|N| = |r_u \times r_v|$ is the area of the parallelogram with sides $r_u$and r_v, by the definition of cross product. Hence

$$n dA = n|N| du dv = N du dv. \tag{1.14}$$

And we see that $dA = |N| du dv$is the element of area of S. Also $F.n$ is the normal component of F. This integral arises naturally in flow problems, where it gives the flux across S when $F = \rho v$. This means, that the flux across S is the mass of fluid crossing S per unit time. Furthermore, ρ is the density of the fluid and v the velocity vector of the flow, as illustrated by Example 1.5 below. We may thus call the surface integral (1.13) the flux integral.

We can write (1.13) in components, using $F = [F_1, F_2, F_3]$, $N = [N_1, N_2, N_3]$, and $n = [\cos\alpha, \cos\beta, \cos\gamma]$. Here, α, β, γ are the angles between n and the

coordinate axes; indeed, for the angle between n and i, which gives $cos\alpha = ni/|n||i| = ni$, $cos\beta = nj/|n||j|$, and so on. We thus obtain from (1.13)

$$\int\int_S FndA = \int\int_S (F_1cos\alpha + F_2cos\beta + F_3cos\gamma) = \int\int_R (F_1N_1 + F_2N_2 + F_3N_3)dudv \tag{1.15}$$

In (1.15) we can write cos a $\cos\alpha dA = dydz$, $\cos\beta dA = dzdx, \cos\gamma dA = dxdy$. Then (1.15) becomes the following integral for the flux:

$$\int\int_S FndA = \int\int_S (F_1dydz + F_2dzdx + F_3dxdy). \tag{1.16}$$

We can use this formula to evaluate surface integrals by converting them to double integrals over regions in the coordinate planes of the xyz-coordinate system. But we must carefully take into account the orientation of S (the choice of n). We explain this for the integrals of the F_3-terms,

$$\int\int_S F_3\cos\gamma dA = \int\int_S F_3dxdy \tag{1.17}$$

If the surface S is given by $z = h(x, y)$ with (x, y) varying in a region R in the xy-plane, and if S is oriented so that $\cos\gamma \geq 0$, then (1.17) gives

$$\int\int_S F_3\cos\gamma dA = \int\int_R F_3(x, y, h(x, y))dxdy. \tag{1.18}$$

But if $\cos\gamma < 0$, the integral on the right of (1.18) gets a minus sign in front. This follows if we note that the element of area $dxdy$ in the xy-plane is the projection $|\cos\gamma|dA$ of the element of area dA of S; and we have $\cos\gamma = |\cos\gamma|$when $\cos\gamma > 0$, but $\cos\gamma = -|\cos\gamma| when \cos\gamma < 0$. Similarly for the other two terms in (1.17). At the same time, this justifies the notations in (1.17).

Example 1.5. *Compute the flux of water through the parabolic cylinder $S : y = x^2,\ 0 \leq x \leq 2,\ 0 \leq z \leq 3$ if the velocity vector is $v = F = [3z^2, 6, 6xz]$, speed being measured in meters per sec. (Generally, $F = \rho v$, but water has the density $\rho = 1g/cm^3 = 1ton/m^3$.)*

Solution

Writing$x = u$and $z = v$, we have $y = x^2 = u^2$. Hence a representation of S is
$S : r[u, u^2, v] \qquad (0 \leq u \leq 2, 0 \leq v \leq 3)$.
By differentiation and by the definition of the cross product
$N = r_u \times r_v = [1, 2u, 0] \times [0, 0, 1] = [2u, 1, 0]$.
On S, writing simply $F(S)$ for $F\,[r(u, v)]$, we have $F(S)[3v^2, 6, 6uv]$. Hence$F(S)N = 6uv^2 - 6$. By rom (3) t integration we thus get

$$\int\int_S FndA = \int_0^3\int_0^2 (6uv^2-6)dudv$$

$$= \int_0^3 (3u^2v^2-6u)|_{u=0}^2 dv = \int_0^3 (12v^2-12)dv = (4v^3-12)|_{v=0}^3 = 108-36 = 72[m^3/sec]$$

Triple Integrals.

A triple integral is an integral of a function $f(x,y,z)$ taken over a closed bounded, three-dimensional region T in space. We subdivide T by planes parallel to the coordinate planes. Then we consider those boxes of the subdivision that lie entirely inside T, and number them from $1\ to\ n$. Here each box consists of a rectangular parallelepiped. In each such box we choose an arbitrary point, say, (x_k,y_k,z_k) in box k. The volume of box $k$we denote by ΔV_k. We now form the sum

$$J_n = \sum_{k=1}^{n} f(x_k,y)_k,z_k)\Delta V_k.$$

This we do for larger and larger positive integers n arbitrarily but so that the maximum length of all the edges of those n boxes approaches zero as n approaches infinity. This gives a sequence of real numbers $Jn_1, Jn_2, \cdot$, We assume that $f(x,y,z)$ is continuous in a domain containing T, and T is bounded by finitely many smooth surfaces . Then it can be shown that the sequence converges to a limit that is independent of the choice of subdivisions and corresponding points (x_k,y_k,z_k). This limit is called the triple integral of $f(x,y,z)$ over the region T and is denoted by

$$\int\int\int_T f(x,y,z)dxdydz$$

or by

$$\int\int\int_T f(x,y,z)dV.$$

Triple integrals can be evaluated by three successive integrations. This is similar to the evaluation of double integrals by two successive integrations

Divergence Theorem of Gauss

Triple integrals can be transformed into surface integrals over the boundary surface of a region in space and conversely. Such a transformation is of practical interest because one of the two kinds of integral is often simpler than the other. It also helps in establishing fundamental equations in fluid flow, heat conduction, etc., as we shall see. The transformation is done by

the divergence theorem, which involves the divergence of a vector function $F = [F_1, F_2, F_3] = F_1 i + F_2 j + F_3 k$, namely,

$$divF = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} \tag{1.19}$$

• Divergence Theorem of Gauss (Transformation Between Triple and Surface Integrals)

Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $F(x, y, z)$ be a vector function that is continuous and has continuous first partial derivatives in some domain containing T. Then

$$\int\int\int_T divF dv = \int\int_S F.ndA. \tag{1.20}$$

In components of $F = [F_1, F_2, F_3]$ and of the outer unit normal vector $n = [\cos\alpha, \cos\beta, \cos\gamma]$ of S , formula (1.19) becomes

$$\int\int\int_T \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right) dxdydz = \int\int_S [F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma]\, dA \tag{1.21}$$

$$= \int\int_S [F_1 dydz + F_2 dzdx + F_3 dxdy]\,.$$

Example 1.6. *Evaluate*

$$\int\int_S \left[x^3 dydz + x^2 ydzdx + x^2 zdxdy\right].$$

where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2 (0 \le z \le b)$ and the circular disks $z = 0$ and $z = b(x^2 + y^2 \le a^2)$.

Solution

$F_1 = x^3, F_2 = x^2 y, F_3 = x^2 z$. **Hence** $divF = 3x^2 + x^2 + x^2 = 5x^2$. **The form of the surface suggest us to introduce polar coordinate**

$$x = r\cos\theta, \;\; y = r\sin\theta \;\; and \;\; dxdydz = rdrd\theta dz$$

we obtain

$$I = \int\int\int_T 5x^2 dxdydz = 5\int_{z=0}^{b}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\cos^2\theta rdrd\theta dz$$

$$= 5b\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\cos^2\theta rdrd\theta = 5b\frac{a^4}{4}\int_{\theta=0}^{2\pi}\cos^2\theta d\theta = \frac{5}{4}\pi ba^4$$

Further Applications of the Divergence Theorem

Potential Theory. Harmonic Functions

The theory of solution of Laplace equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{1.22}$$

is called potential theory. A solution of (1.22) **with continuous second-order partial derivatives is called a harmonic function. That continuity is needed for application of the divergence theorem in potential theory, where the theorem plays a key role**

Example 1.7. *The integrands in the divergence theorem are* $divF$ *and* Fn *. If* F *is the gradient of a scalar function, say,* $F = gradf$*, then* $divFdiv(gradf) = \nabla^2 f;$*. Also,* $Fn = nF = ngradf$*. This is the directional derivative of* f *in the outer normal direction of* S*, the boundary surface of the region* T *in the theorem. This derivative is called the (outer) normal derivative of* f *and is denoted by* $\frac{\partial f}{\partial n}$*. Thus the formula in the divergence theorem becomes*

$$\int\int\int_T \nabla^2 f dv = \int\int_S \frac{\partial f}{\partial n} dA$$

• **Let** $f(x, y, z)$ **be a harmonic function in some domain** D **is space. Let** S **be any piecewise smooth closed orientable surface in** D **whose entire region it encloses belongs to** D**. Then the integral of the normal derivative of** f **taken over** S **is zero**

• **Let** $f(x, y, z)$ **be harmonic in some domain** D **and zero at every point of a piecewise smooth closed orientable surface** S **in** D **whose entire region** T **it encloses belongs to** D**. Then** f **is identically zero in** T**.**

• **Let T be a region that satisfies the assumptions of the divergence theorem, and let** $f(x, y, z)$ **be a harmonic function in a domain** D **that contains** T **and its boundary surface** S **. Then** f **is uniquely determined in** T **by its values on** S **.**

• **If the above assumptions are satisfied and the Dirichlet problem for the Laplace equation has a solution in T, then this solution is unique.**

Stokes Theorem (Transformation Between Surface and Line Integrals) **Let** S **be a piecewise smooth oriented surface in space and**

let the boundary of S be a piecewise smooth simple closed curve C. Let $F(x,y,z)$ be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

$$\int\int_S (curlF).ndA = \oint_C F.r'(s)ds \tag{1.23}$$

Here n is a unit normal vector of S and, depending on n, the integration around C . Furthermore, $r' = \frac{dr}{ds}$ is the unit tangent vector and s the arc length of C
In component form formula (1.23) can be written as

$$\int\int_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right] dudv \tag{1.24}$$
$$= \oint_{\bar{C}} (F_1dx + F_2dy + F_3dz)$$

where R is the region of the curve $\bar{C}$ in $uv-$plane corresponding to S represented by $r(u,v)$ and $N = [N_1, N_2, N_3] = r_u \times r_v$

Example 1.8. *In this example we verify Stokes theorem*
Given $F = [y,z,x] = yi + zj + xk$ and S is the paraboloid $z = f(x,y) = 1 - (x^2 + y^2)$ $z \geq 0$

Solution
The curve C is the circle $r(s) = [\cos s, \sin s, 0] = \cos si + \sin sj$ it has the unit tangent vector $r'(s) = [-\sin s, \cos s, 0] = -\sin si + \cos sj$. Then the line integral in (1.23) on the right

$$\oint_C F.dr = \int_0^{2\pi} [(\sin s)(-\sin s) + 0 + 0]\, ds = -\pi$$

Left side of (1.23)
$curlF = [-1,-1,-1]$ *and* $N = grad(z - f(x,y)) = [2x, 2y, 1]$ **so that** $curlF.N = -2x - 2y - 1$

$$\int\int_S curlF.ndA = \int\int_R (2x - 2y - 1)dxdy$$
$$= \int\int_R (-2r\cos\theta - -2r\sin\theta - 1)rdrd\theta$$
$$= \int_0^{1`} \int_0^{2\pi} (-2r\cos\theta - -2r\sin\theta - 1)rdrd\theta$$
$$= 0 + 0 + (\frac{-1}{2})2\pi = -\pi$$

Assignment-I
Evalute the following integral and check for path independence.

1 $F[y^2, x^2], C: y = 4x^2\ from\ (0,0)\ to\ (1,4)$
2 $F[xy, x^2y^2], C\ from\ (2,0)\ straight\ to\ (0,2)$
3 F as in Prob. 2, C the quarter-circle from (2, 0) to (0, 2) with center (0, 0)
4 $F = [xy, yz, zx], C: r(t) = [2cost, t, 2sint]\ from\ (2,0,0)\ to\ (2, 2\pi, 0)$
5 $F = [x^2, y^2, z^2], C: r(t) = [cost, sint, e^t]\ from\ (1,0,1) to (1,0,e^2\pi)$**.**
6 $F = [x, z, 2y]\ from\ (0,0,0)\ straight\ to (1,1,0),\ then\ to\ (1,1,1),\ back\ to\ (0,0,0)$
7

$$\int_0^3 \int_{-y}^{y} (x^2 + y^2) dxdy$$

Evaluate line integral using Green;s theorem

8

$$F = \left[x^2 e^y, y^2 e^x\right]$$

C**: is the rectangle with vertices (0,0)(2,0)(2,3)(0,3)**
9

$$F = [y, -x]$$

C**: is the circle** $x^2 + y^2 = \frac{1}{4}$
10 Find parametric representation of the surface

$$x^2 + y^2 + \frac{1}{4} z^2 = 1$$

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LAPLACE TRANSFORMS

Laplace transforms are very important for any engineers mathematical toolbox as they make solving linear Ordinary Differential Equations and related initial value problems, as well as systems of linear ODEs, much easier. Applications abound: electrical networks, springs, mixing problems, signal processing, and other areas of engineering and physics. The process of solving an ODE using the Laplace transform method consists of three steps, :
Step 1. The given ODE is transformed into an algebraic equation, called the subsidiary equation.
Step 2. The subsidiary equation is solved by purely algebraic manipulations.
Step 3. The solution in Step 2 is transformed back, resulting in the solution of the given problem.
IVP Initial Value Problem$\rightarrow$ AP Algebraic Problem $\rightarrow$ Solving AP by Algebra$\rightarrow$ Solution of the IVP.

This Solving an IVP by Laplace transforms

The key motivation for learning about Laplace transforms is that the process of solving an ODE is simplified to an algebraic problem (and transformations). This type of mathematics that converts problems of calculus to algebraic problems is known as operational calculus. The Laplace transform method has two main advantages over the methods discussed earlier: I. Problems are solved more directly: Initial value problems are solved without first determining a general solution. Non-homogenous ODEs are solved without first solving the corresponding homogeneous ODE. II. More importantly, the use of the unit step function (Heaviside function) and Diracs delta function make the method particularly powerful for problems with inputs (driving forces) that have discontinuities or represent short impulses or complicated periodic functions.

1 Laplace Transform. Linearity. First Shifting Theorem (s-Shifting)

In this section, we learn about Laplace transforms and some of their properties. Because Laplace transforms are of basic importance to the engineer, the student should pay close attention to the material. Applications to ODEs follow in the next section. Roughly speaking, the Laplace transform, when applied to a function, changes that function into a new function by using a process that involves integration. Details are as follows.

If $f(t)$ is a function defined for all $t \geqq 0$, its Laplace transform1 is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. It is a function of s, say, $F(s)$, and is denoted by $\mathcal{L}(f)$: thus

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t) dt \tag{1}$$

Here we must assume that $f(t)$ is such that the integral exists (that is, has some finite value). This assumption is usually satisfied in applicationswe shall discuss this near the end of the section.

Not only is the result $F(s)$ called the Laplace transform, but the operation just described, which yields $F(s)$from a given $f(t)$, is also called the Laplace transform. It is an integral transform

$$F(s) = \int_0^\infty k(s,t) f(t) dt$$

with kernel $k(s,t) = e^{-st}$. Note that the Laplace transform is called an integral transform because it transforms (changes) a function in one space to a function in another space by a process of integration that involves a kernel. The kernel or kernel function is a function of the variables in the two spaces and defines the integral transform.

Furthermore, the given function $f(t)$ in (1) is called the inverse transform of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$; that is, we shall write

$$f(t) = \mathcal{L}^{-1}(F) \tag{2}$$

Note that (1) and (2) together imply and $\mathcal{L}^{-1}(\mathcal{L}(F))$ and $\mathcal{L}(\mathcal{L}^{-1}(F)) = F$.

E X A M P L E 1 Laplace Transform Let $f(t) = 1$ when $t \geqq 0$. Find $F(s)$.

Solution. From (1) we obtain by integration

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st} dt = \frac{1}{s}$$

Such an integral is called an improper integral and, by definition, is evaluated according to the rule

$$\int_0^\infty e^{-st} f(t) dt = \lim_{T \to \infty} \int_0^T e^{-st} f(t) dt$$

Hence our convenient notation means

$$\int_0^\infty e^{-st} dt = \lim_{T \to \infty} \int_0^T e^{-st} dt = \frac{1}{s}.$$

We shall use this notation throughout this chapter.

E X A M P L E 2 Laplace Transform $\mathcal{L}(e^{at})$ of the Exponential Function e^{at}

Let $f(t) = e^{at}$ when $t \geqq 0$, where a is a constant. Find $F(s)$.

Solution. Again by (1)

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st} e^{at} dt = \frac{1}{s} = \frac{1}{s}.$$

hence, when $s - a > 0$,

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}.$$

Must we go on in this fashion and obtain the transform of one function after another directly from the definition? No! We can obtain new transforms from known ones by the use of the many general properties of the Laplace transform. Above all, the Laplace transform is a linear operation, just as are differentiation and integration. By this we mean the following.

Theorem 1.1. ***Linearity of the Laplace Transform*** *The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and*

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + a\mathcal{L}\{g(t)\}$$

Proof. This is true because integration is a linear operation so that (1) gives

$$\begin{aligned}\mathcal{L}\{af(t)+bg(t)\} &= \int_0^\infty e^{-st}\{af(t)+bg(t)\}dt \\ &= a\int_0^\infty e^{-st}\{af(t)\}+b\int_0^\infty e^{-st}\{g(t)\} \\ &= a\mathcal{L}\{f(t)\}+a\mathcal{L}\{g(t)\}\end{aligned}$$

□

Example:Application of Theorem 1: Hyperbolic Functions Find the transforms of $\cos hat$ and $\sin hat$.
Solution Since $\cos hat = \frac{1}{2}(e^{at}+e^{-at})$ and $\sin hat = \frac{1}{2}(e^{at}-e^{-at})$ we obtain from Example 2 and Theorem 1

$$\mathcal{L}\{\cos hat\} = \frac{1}{2}\mathcal{L}\{e^{at}\}+\frac{1}{2}\mathcal{L}\{e^{-at}\} = \frac{s}{s^2-a^2}.$$

$$\mathcal{L}\{\sin hat\} = \frac{1}{2}\mathcal{L}\{e^{at}\}-\frac{1}{2}\mathcal{L}\{e^{-at}\} = \frac{a}{s^2-a^2}.$$

Example:Cosine and Sine
Find the transforms of $\cos \omega t$ and $\sin \omega t$.
Solution We write $L_c = \mathcal{L}\cos \omega t$and $L_s = \mathcal{L}\sin \omega t$. Integrating by parts and noting that the integral free parts give no contribution from the upper limit ∞, we obtain

$$L_c = \mathcal{L}\{\cos \omega t\} = \int_0^\infty e^{-st}\{\cos \omega t\}dt = \frac{s}{s^2+\omega^2}$$
$$L_s = \mathcal{L}\{\sin \omega t\} = \int_0^\infty e^{-st}\{\sin \omega t\}dt = \frac{\omega}{s^2+\omega^2}.$$

Basic transforms are listed in the folloing Table. We shall see that from these almost all the others can be obtained by the use of the general properties of the Laplace transform. Formulas 13 are special cases of formula 4, which is proved by induction. Indeed, it is true for $n = 0$ because of Example 1 and $0! = 1$. We make the induction hypothesis that it holds for any integer $n \geq 0$ and then get it for $n+1$ directly from (1). Indeed, integration by parts first gives

$$\mathcal{L}(t^{n+1}) = \int_0^\infty e^{-st} t^{n+1} dt$$

Now the integral-free part is zero and the last part is $(n+2)/s$ times $\mathcal{L}$. From this and the induction hypothesis,

$$\mathcal{L}(t^{n+1}) = \frac{(n+1)!}{s^{n+2}}$$

No	$f(t)$	$L\{f\}$	No	$f(t)$	$L\{f\}$
1	1	$\frac{1}{s}$	7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
2	t	$\frac{1}{s^2}$	8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
3	t^2	$\frac{2!}{s^3}$	9	$\cos hat$	$\frac{s}{s^2-a^2}$
4	t^n	$\frac{n!}{s^{n+1}}$	10	$\sin hat$	$\frac{a}{s^2-a^2}$
5	t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

2 s-Shifting: Replacing s by s-a in the Transform

The Laplace transform has the very useful property that, if we know the transform of $f(t)$ we can immediately get that of $e^{at}f(t)$ as follows.

Theorem 2.1. ***First Shifting Theorem, s-Shifting*** *If $f(t)$ has the transformation $F(s)$ where $s > k$ for some k, the $e^{at}f(t)$ has the transforms $F(s-a)$ (where $s-a > k$). In formulas*

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

or, if we take the inverse on both sides,

$$\{e^{at}f(t)\} = \mathcal{L}^{-1}\{F(s-a)\}$$

Proof. We obtain $F(s-a)$ by replacing s with $s-a$ in the integral in (1), so that

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t) dt = \int_0^\infty [e^{-st}[e^{at}f(t)]dt = \mathcal{L}\{e^{at}f(t)\}.$$

If F(s) exists (i.e., is finite) for s greater than some k, then our first integral exists for $s - a > k$. Now take the inverse on both sides of this formula to obtain the second formula in the theorem. □

Examples-Shifting: Damped Vibrations. Completing the Square From Example 4 and the first shifting theorem we immediately obtain formulas 11 and 12 in Table 1,

$$\mathcal{L}\{\cos \omega t\} = \frac{s-a}{(s-a)^2 + \omega^2}$$

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{(s-a)^2 + \omega^2}$$

For instance, use these formulas to find the inverse of the transform

$$\mathcal{L}\{f\} = \frac{3s - 137}{s^2 + 2s + 401}$$

Solution

Applying the inverse transform, using its linearity, and completing the square, we obtain

$$f = \mathcal{L}^{-1}\{\frac{3(s+1) - 140}{s + 1^2 + 400}\} - 3\mathcal{L}^{-1}\{\frac{(s+1)}{s + 1^2 + 20^2}\} - 7\mathcal{L}^{-1}\{\frac{20}{s + 1^2 + 20^2}\}$$

We now see that the inverse of the right side is the damped vibration

$$f(t) = e^{-t}(3\cos 20t - 7\sin 20t).$$

Home Work

Find the inverse Laplace Transform of the following functions.

1. $\frac{e^{-3s}}{(s-1)^2}$
2. $\frac{e^{-3s}}{s^3}$
3. $\frac{4(e^{-2s} - 2e^{-5s})}{s}$
4. $\frac{e^{-2\pi s}}{s^2+2s+2}$
5. $\frac{3(1-e^{-\pi s})}{(s^2+9)}$
6. $\frac{se^{-2s}}{(s^2+\pi^2)}$

3 Existence and Uniqueness of Laplace Transforms

This is not a big practical problem because in most cases we can check the solution of an ODE without too much trouble. Nevertheless we should be aware of some basic facts. A function $f(t)$ has a Laplace transform if it does not grow too fast, say, if for all $t \geqq 0$ and some constants M and k it satisfies the growth restriction

$$|f(t)| \leq Me^{kt} \tag{3}$$

(The growth restriction (2) is sometimes called growth of exponential order, which may be misleading since it hides that the exponent must be kt, not kt^2 or similar.) $f(t)$ need not be continuous, but it should not be too bad. The technical term (generally used in mathematics) is piecewise continuity. $f(t)$ is piecewise continuous on a finite interval $a \leq t \leq b$ where f is defined, if this interval can be divided into finitely many subintervals in each of which f is continuous and has finite limits as t approaches either endpoint of such a subinterval from the interior. This then gives finite jumps as the only possible discontinuities, but this suffices in most applications, and so does the following theorem.

Theorem 3.1. *If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and satisfies $t \geq 0$ and satisfies* (3) *for all $t \geq 0$ and some constants M and k, then the Laplace transform $\mathcal{L}(f)$ exists for all s¿k.*

Proof. Since $f(t)$ is piecewise continuous, $e^{-st}f(t$ is integrable over any finite interval on the t-axis. From (3), assuming that ($s > k$ to be needed for the existence of the last of the following integrals), we obtain the proof of the existence of $\mathcal{L}(f)$ from

$$|\mathcal{L}(f)| = |\int_0^\infty e^{-st}f(t)dt| \leq \int_0^\infty e^{-st}|f(t)|dt \leq \int_0^\infty e^{-st}Me^{kt}dt = \frac{M}{s-k}$$

Note that (3) can be readily checked. For instance, $\cosh t < e^t$, $t^n < n!e^t$ (because $t^n/n!$ is a single term of the Maclaurin series), and so on. A function that does not satisfy (3) for any M and k is e^{t^2} (take logarithms to see it). We mention that the conditions in Theorem 3 are sufficient rather than necessary. Uniqueness. If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points. Hence we may say that the inverse of a given transform is essentially unique. In particular, if two continuous functions have the same transform, they are completely identical. □

4 Transforms of Derivatives and Integrals. ODEs

The Laplace transform is a method of solving ODEs and initial value problems. The crucial idea is that operations of calculus on functions are replaced by operations of algebra on transforms. Roughly, differentiation of $f(t)$ will correspond to multiplication of by s (see Theorems 1 and 2) and integration of $f(t)$ to division of $\mathcal{L}(f)$ by s. To solve ODEs, we must first consider the Laplace transform of derivatives. You have encountered such an idea in your study of logarithms. Under the application of the natural logarithm, a product of numbers becomes a sum of their logarithms, a division of numbers becomes their difference of logarithms. To simplify calculations was one of the main reasons that logarithms were invented in pre-computer times.

Theorem 4.1. ***Laplace Transform of Derivatives*** *The transforms of the first and second derivatives of $f(t)$ satisfy*

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \tag{4}$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f.(0) \tag{5}$$

Formula (4) *holds if $f(t)$ is continuous for all $t \geq 0$ and satisfies the growth restriction* (3) *in the previous section and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geq 0$. Similarly,* (5) *holds if f and f' are continuous for all $t \geq 0$ and satisfy the growth restriction and f'' is piecewise continuous on every finite interval on the semi-axis $t \geq 0$.*

Proof. We prove (4) first under the additional assumption that f' is continuous. Then, by the definition and integration by parts,

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)dt = [e^{-st} f(t)]|_0^\infty dt = s \int_0^\infty e^{-st} ftdt$$

Since f satisfies (5) , the integrated part on the right is zero at the upper limit when $s > k$ and at the lower limit it contributes $-f(0)$. The last integral is $\mathcal{L}(f)$. It exists for $s > k$ because of Theorem 3. Hence $\mathcal{L}(f')$ exists when $s > k$ and (4) holds. If f' is merely piecewise continuous, the proof is similar. In this case the interval of integration of f' must be broken up into parts such that f' is continuous in each such part.

The proof of (3) now follows by applying (4) to f'' and then substituting (4), that is

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s[s\mathcal{L}(f) - f(0)] = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

Continuing by substitution as in the proof of (2) and using induction, we obtain the following extension of Theorem 1. □

Theorem 4.2. ***Laplace Transform of the Derivative*** f^n ***of Any Order*** *Let* $f, f', f'', \cdots, f^{n-1}$ *be continuous for all* $t \geq 0$

and satisfy the growth restriction (5). *Furthermore, let* $f^{(n)}$ *be piecewise continuous on every finite interval on the semi-axis* $t \geq 0$. *Then the transform of* $f^{(n)}$ *satisfies*

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - f^{(n-1)}(0) \tag{6}$$

Example 4.3.

Transform of a Resonance Term Let $f(t) = t \sin \omega t$. Then $f(t) = 0,\ \ f'(t) = \sin \omega t + \omega t \cos \omega t,\ \ f'(0) = 0,\ \ f'' = 2\omega \cos \omega t - \omega^2 t \sin \omega t$. Hence by (6)

$$\mathcal{L}(f'') = 2\omega \frac{s}{s^2 + \omega^2} - \omega^2 \mathcal{L}(f) = s^2\mathcal{L}(f).$$

Thus

$$\mathcal{L}(f) = \mathcal{L}(t \sin \omega t) = \frac{2\omega s}{(s^2 + \omega)^2}$$

Example 4.4.

This is a third derivation of $\mathcal{L}(\cos\omega t)$ and $\mathcal{L}(\sin\omega t)$. Let $f(t) = \cos\omega t$, then

$$f(0) = 1, \quad f'(0) = 0, \quad f''(t) = -\omega^2\cos\omega t.$$

From this and (5) we obtain

$$\mathcal{L}(f'') - s^2\mathcal{L}(f) - s = -\omega^2\mathcal{L}(f).$$

By algebra

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}.$$

Similarly, let $g = \sin\omega t$. Then $g(0) = 0, \;\; g' = \omega\cos\omega t$. From this and (4) we obtain

$$\mathcal{L}(g') = s\mathcal{L}(g) = \omega\mathcal{L}(\cos\omega t).$$

Hence,

$$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}.$$

5 Laplace Transform of the Integral of a Function

Differentiation and integration are inverse operations, and so are multiplication and division. Since differentiation of a function $f(t)$ (roughly) corresponds to multiplication of its transform $\mathcal{L}(f)$ by s, we expect integration of $f(t)$ to correspond to division of $\mathcal{L}(f)$ by s:

Theorem 5.1. ***Laplace Transform of Integral*** *Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geqq 0$ and satisfies a growth restriction* (5). *Then, for $s > 0, \;\; s > k$ and $t > 0$,*

$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}F(s), \quad \int_0^t f(\tau)d\tau = \mathcal{L}^{-1}\{\frac{1}{s}F(s)\} \tag{7}$$

Proof. Denote the integral in (7) by $g(t)$. Since $f(t)$ is piecewise continuous, $g(t)$ is continuous, and (5) gives

$$|g(t) = |\int_0^t f(\tau)d\tau| \le \int_0^t |f(\tau)|d\tau \le M\int_0^t e^{kt}d\tau = \frac{M}{k}(e^{kt}-1) \le \frac{M}{k}(e^{kt}), \;\; k > 0.$$

This shows that $g(t)$ also satisfies a growth restriction. Also, $g'(t) = f(t)$ except at points at which $f(t)$ is discontinuous. Hence $g'(t)$ is piecewise continuous on each finite interval and, by Theorem 1, since $g(t) = 0$(the integral from 0 to 0 is zero)

$$\mathcal{L}\{(f(t))\} = \mathcal{L}(g'(t) = s\mathcal{L}(g(t))) - g(0) = s\mathcal{L}(g(t))$$

Division by s and interchange of the left and right sides gives the first formula in (4), from which the second follows by taking the inverse transform on both sides. □

Example 5.2.
Application of Theorem (5.1)**:**

Using theorem 3, find the inverse of $\frac{1}{s(s^2+\omega^2)}$ **and** $\frac{1}{s^2(s^2+\omega^2)}$

solution From Table and the integration in (7) (second formula with the sides interchanged) we obtain

$$\mathcal{L}^{-1}\{\frac{1}{s^2+\omega^2}\} = \frac{\sin\omega t}{s}, \quad \mathcal{L}^{-1}\{\frac{1}{s(s^2+\omega^2)}\} = \int_0^t \frac{\sin\omega t}{\omega} d\omega == \frac{1}{\omega^2}(1-\cos\omega t)$$

Integrating this result again and using (7) as before, we obtain

$$\mathcal{L}^{-1}\{\frac{1}{s^2(s^2+\omega^2)}\} = \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)d\tau = \frac{t}{\omega^2} - \frac{\sin\omega\tau}{\omega^3}$$

It is typical that results such as these can be found in several ways. In this example, try partial fraction reduction.

6 Differential Equations, Initial Value Problems

Let us now discuss how the Laplace transform method solves ODEs and initial value problems. We consider an initial value problem

$$y'' + ay' + by = r(t), \;\; y(0) = K_0, \;\; y'(0) = K_1 \tag{8}$$

where a and b are constant. Here $r(t)$ is the given input (driving force) applied to the mechanical or electrical system and $y(t)$ is the output (response to the input)

to be obtained. In Laplaces method we do three steps:
Step 1. Setting up the subsidiary equation. This is an algebraic equation for the transform $Y = \mathcal{L}(y)$ obtained by transforming (8) by means of (1) and (3), namely,

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $R(s)\mathcal{L}(r)$. Collecting the Y-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

Step 2. Solution of the subsidiary equation by algebra. We divide by $s^2 + as + b$ and use the so-called transfer function

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2} \tag{9}$$

(Q is often denoted by H, but we need H much more frequently for other purposes.) This gives the solution

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s) \tag{10}$$

If $y(0) = y'(0) = 0$, this imply $Y = RQ$: hence

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(output)}{\mathcal{L}(input)}$$

and this explains the name of Q. Note that Q depends neither on r(t) nor on the initial conditions (but only on a and b).
Step 3. Inversion of Y to obtain $y = \mathcal{L}^{-1}(Y)$

We reduce (10) (usually by partial fractions as in calculus) to a sum of terms whose inverses can be found from the tables, so that we obtain the solution $y(t) = \mathcal{L}^{-1}(Y)$ of (8).

Example 6.1.
Initial Value Problem: The Basic Laplace Steps Solve

$$y'' - y = t \;\; y(0) = 1 \;\; y'(0) = 1.$$

Solution

Step 1. From (3) and Table we get the subsidiary equation $[y = \mathcal{L}(Y)]$

$$s^2Y - sy(0) - y'(0) - Y = \frac{1}{s^2},$$

thus

$$(s^2 - 1)Y = S + 1 + \frac{1}{s^2}$$

Step 2. The transfer function is $Q = 1/(s^2 - 1)$, and (10) becomes

$$Y = (s+1)q + \frac{1}{s^2}Q = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}$$

Simplification of the first fraction and an expansion of the last fraction gives

$$Y = \frac{1}{s-1} + (\frac{1}{s^2-1} - \frac{1}{s^2}).$$

Step 3. From this expression for Y and Table 6.1 we obtain the solution

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}(\frac{1}{s-1}) + \mathcal{L}^{-1}(\frac{1}{s^2-1}) - \mathcal{L}^{-1}(\frac{1}{s^2}) = e^t + \sinh t - t.$$

Example 6.2.

Comparison with the Usual Method Solve the initial value problem

$$y'' + y' + 9y = 0, \ \ y(0) = 0.16, , \ \ y'(0) = 0$$

Solution From (1) and (3) we see that the subsidiary equation is

$$s^2Y - 0.16s + sY - 0.16 + 9Y = 0, , \ thus \ (s^2 + s + 9)Y = 0.16(s+1).$$

The solution is

$$Y = \frac{0.16(s+1)}{s^2+s+9} = \frac{0.16(s+\frac{1}{2}) + 0.08}{(s+\frac{1}{2})^2 + \frac{35}{4}}$$

Hence by the first shifting theorem and the formulas for cos and *sin* in Table-1 we obtain

$$\begin{aligned} y(t) = \mathcal{L}^{-1}(Y) \qquad &= e^{\frac{-t}{2}}(0.16\cos\sqrt{\frac{35}{4}}t + \frac{0.08}{\frac{1}{2}\sqrt{35}}\sin\frac{35}{4}t) \\ &= e^{-0.5t}(0.16\cos 2.96t + 0.027\sin 2.96t) \end{aligned}$$

Advantages of the Laplace Method

1. Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.

2. Initial values are automatically taken care of.

3. Complicated inputs $r(t)$ (right sides of linear ODEs) can be handled very efficiently,

Example 6.3.
Shifted Data Problems This means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$. For such a problem set $t = \widetilde{t} + t_0$ so that $t = t_0$ gives $\widetilde{t} = 0$ and the Laplace transform can be applied. For instance, solve

$$y'' + y = 2t, \ \ y(\frac{1}{4}\pi) = \frac{1}{2}\pi, \ \ y'(\frac{1}{4}\pi)2 - \sqrt{2}.$$

7 Unit Step Function (Heaviside Function), Second Shifting Theorem (t-Shifting), Dirac's Delta Function

This section is extremely important because we shall now reach the point where the Laplace transform method shows its real power in applications and its superiority over the classical approach. The reason is that we shall introduce two auxiliary functions, the unit step function or Heaviside function $u(t-a)$(below) and Diracs delta $\delta(t-a)$. These functions are suitable for solving ODEs with complicated right sides of considerable engineering interest, such as single waves, inputs (driving forces) that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive forces.

Unit Step Function (Heaviside Function)
The unit step function or Heaviside function $u(t-a)$ is 0 for $t < a$ has a jump of size 1 at $t = a$ (where we can leave it undefined), and is 1 for $t > a$ in a formula:

$u(t{-}a) = \begin{cases} 0, & \text{if } t < a; \\ 1, & \text{if } t > a. \end{cases}$ The transform of follows directly from the defining integral

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)dt = \int_0^\infty e^{-st}.1\ dt = \left[-\frac{e^{-st}}{s}\right]_{t=a}^{\infty};$$

here the integration begins at $t = a(\geqq 0)$ because $u(t-a)$ is 0 for $t < a$. Hence

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s} \qquad (s > 0) \tag{11}$$

The unit step function is a typical engineering function made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either off or on. Multiplying functions $f(t)$ with $u(t-a)$, we can produce all sorts of effects. More generally we have the following.

Let $f(t) = 0$ for all negative t. Then $f(t-a)u(t-a)$ with $a > 0$ is $f(t)$ shifted (translated) to the right by the amount a.

Time Shifting (t-Shifting): Replacing t by $t - a$ in $f(t)$

The first shifting theorem (s-shifting) concerned transforms $F(s) = \mathcal{L}\{f(t)\}$ and $F(s-a) = \mathcal{L}\{e^{at}f(t)\}$. The second shifting theorem will concern functions $f(t)$ and $f(t-a)$. Unit step functions are just tools, and the theorem will be needed to apply them in connection with any other functions.

Second Shifting Theorem; Time Shifting

Theorem 7.1. *If $f(t)$ has the transform $F(s)$ then the shifted function*

$$\widetilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0, & \text{if } t < a; \\ f(t\text{-}a), & \text{if } t > a. \end{cases}$$

has the transform $e^{-as}F(s)$. That is, if $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s).$$

Or, if we take the inverse on both sides, we can write

$$f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}.$$

Application of Both Shifting Theorems. Inverse Transform

Example 7.2.
Find the inverse transform $f(t)$ of

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}$$

Solution 7.3. Without the exponential functions in the numerator the three terms of $F(s)$ would have the inverses $\frac{\sin \pi t}{\pi}$, $\frac{\sin \pi t}{\pi}$ and te^{-2t} because $\frac{1}{s^2}$ has the inverse t, so that $\frac{1}{(s+2)^2}$ has the inverse te^{-2t} by the first shifting theorem. Hence by the second shifting theorem (t-shifting),

$$f(t) = \frac{1}{\pi}\sin(\pi(t-1))u(t-1) + \frac{1}{\pi}\sin(\pi(t-2))u(t-2) + (t-3)e^{-2(t-3)}u(t-3).$$

Now $\sin(\pi t - \pi) = -\sin \pi t$ and $\sin(\pi t - 2\pi) = \sin \pi t$, so that the first and second terms cancel each other when $t > 2$. Hence we obtain $f(t) = 0$ if $0 < t < 1$, $\frac{-\sin \pi t}{\pi}$ if $1 < t < 2, 0$ if $2 < t < 3$, and $(t-3)e^{-2(t-3)}$ if $t > 3$.

Home Work

Using the Laplace transform and showing the details, solve

1. $9y'' - 6y' + y = 0, y(0) = 3, y'(0) = 1$
2. $y'' + 6y' + 8y = e^{-3t} - e^{-5t}, y(0) = 0, y'(0) = 0.$
3. $y'' + 10y' + 24y = 144t^2, y(0) = \frac{19}{12}, y'(0) = -5$
4. $y'' + 3y' + 2y = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $y(0) = 0, y'(0) = 0$
5. $y'' + y = t$ if $0 < t < 1$ and 0 if $t > 1$; $y(0) = 0, y'(0) = 0.$

8 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

The variety of methods for obtaining transforms and inverse transforms and their application in solving ODEs is surprisingly large. We have seen that they include direct integration, the use of linearity, shifting, convolution, and differentiation and integration of functions $f(t)$. In this section, we shall consider operations of somewhat lesser importance. They are the differentiation and integration of transforms $F(s)$ and corresponding operations for functions $f(t)$. We show how they are applied to ODEs with variable coefficients.

Differentiation of Transforms It can be shown that, if a function f(t) satisfies the conditions of the existence theorem in Sec. 6.1, then the derivative $F'(s) = \frac{dF}{ds}$ of the transform $F(s) = \mathcal{L}(f)$ can be obtained by differentiating $F(s)$ under the integral sign with respect to s. Thus, if

$$F(s) = \int_0^\infty e^{-st} f(t)dt,$$

then

$$F'(s) = -\int_0^\infty e^{-st} t f(t)dt.$$

Consequently, if $\mathcal{L}(f) = F(s)$, then

$$\mathcal{L}\{tf(t)\} = -F'(s), \ \ hence \ \ -tf(t) = \mathcal{L}^{-1}\{F'(s)\}, \tag{12}$$

where the second formula is obtained by applying $\mathcal{L}^{-1}$ on both sides of the first formula. In this way, differentiation of the transform of a function corresponds to the multiplication of the function by $-t$.

Example 8.1.

Differentiation of Transforms. We shall derive the following three formulas.

$\frac{1}{(s^2+\beta^2)^2}$	$\frac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)$
$\frac{s}{(s^2+\beta^2)^2}$	$\frac{1}{2\beta}(\sin\beta t)$
$\frac{s^2}{(s^2+\beta^2)^2}$	$\frac{1}{2\beta^3}(\sin\beta t + \beta t\cos\beta t)$

Solution From (12) and formula 8 (with $\omega = \beta$) in Table 6.1 of Sec. 6.1 we obtain by differentiation

$$\mathcal{L}(t\sin\beta t) = \frac{2\beta s}{(s^2+\beta^2)^2}$$

Dividing by 2β and using the linearity of , we obtain from the table.

$$\mathcal{L}(t\cos\beta t) = -\frac{(s^2+\beta^2) - 2s^2}{(s^2+\beta^2)^2} = \frac{(s^2-\beta^2)}{(s^2+\beta^2)^2} \tag{13}$$

From the forst table we have

$$\mathcal{L}(t\cos\beta t \pm \frac{1}{\beta}\sin\beta t) = \frac{(s^2-\beta^2)}{(s^2+\beta^2)^2} \pm \frac{1}{(s^2+\beta^2)^2}$$

On the right we now take the common denominator. Then we see that for the plus sign the numerator becomes $s^2+\beta^2+s^2+\beta^2 = 2s^2$, so that (4) follows by division by 2. Similarly, for the minus sign the numerator takes the form $s^2-\beta^2-s^2-\beta^2 = -2\beta^2$.

Integration of Transforms Similarly, if f(t) satisfies the conditions of the existence theorem and the limit of $\frac{f(t)}{t}$, as t approaches 0 from the right, exists, then for $s > k$,

$$\mathcal{L}\{\frac{f(t)}{t}\} = \int_s^\infty F(\tilde{s})d\tilde{s}, \;\; hence \;\; \mathcal{L}^{-1}\{\int_s^\infty F(\tilde{s})d\tilde{s}\} = \{\frac{f(t)}{t}\} \qquad (14)$$

In this way, integration of the transform of a function f(t) corresponds to the division of f(t) by t. We indicate how (14)is obtained. From the definition it follows

that

$$\int_s^\infty F(\tilde{s})d\tilde{s} = \int_s^\infty [\int_0^\infty e^{(-\tilde{s}t)}d\tilde{s}]dt$$

Integration of $e^{(-\tilde{s}t)}$ with respect to $(\tilde{s}t)$ gives $e^{(-\tilde{s}t)/(-t)}$. Here the integral over on the right equals $e^{(-\tilde{s}t)/t}$. Therefore,

$$\int_s^\infty F(\tilde{s})d\tilde{s} = \int_0^\infty e^{(-\tilde{s}t)}\frac{f(t)}{t}dt = \mathcal{L}\{\frac{f(t)}{t}\}, \;\; s > k$$

Example 8.2.
Differentiation and Integration of Transforms Find the inverse transform of $\ln(1+\frac{\omega^2}{s^2}) = \ln\frac{s^2+\omega^2}{s^2}$

Solution 8.3. Denote the given transform by F(s). Its derivative is

$$F'(s) = \frac{d}{ds}(\ln(s^2+\omega^2) - \ln s^2) = \frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}$$

Taking the inverse transform and using (12), we obtain

$$\mathcal{L}^{-1}\{F'(s)\} = \mathcal{L}^{-1}\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\} = 2\cos\omega t - 2 = -tf(t).$$

Hence the inverse $f(t)$ of $F(s)$ is $f(t) = 2(1 - \cos \omega t)/t$. Alternatively, if we let

$$G(s) = \frac{2s}{s^2 + \omega^2} - \frac{2}{s}, \quad then \quad g(t) = \mathcal{L}^{-1}(G) - 2(\cos \omega t - 1)$$

Find this and (14) we get, in agreement with the answer just obtained,

$$\mathcal{L}^{-1}\{\ln \frac{s^2 + \omega^2}{s^2}\} = \mathcal{L}^{-1}\{\int_0^\infty G(s)ds\} = \frac{-g(t)}{t} = \frac{2}{t}(1 - \cos \omega t)$$

the minus occurring since s is the lower limit of integration. In a similar way we obtain

$$\mathcal{L}^{-1}\{\ln(1 - \frac{a^2}{s^2})\} = \frac{2}{t}(1 - \cosh at)$$

9 Convolution. Integral Equations

Convolution has to do with the multiplication of transforms. The situation is as follows. Addition of transforms provides no problem; we know that $\mathcal{L}(f + g) = \mathcal{L}(f) + \mathcal{L}(f)$. Now multiplication of transforms occurs frequently in connection with ODEs, integral equations, and elsewhere. Then we usually know $\mathcal{L}(f)$ and $\mathcal{L}(g)$ and would like to know the function whose transform is the product $\mathcal{L}(f)\mathcal{L}(g)$. We might perhaps guess that it is fg, but this is false. The transform of a product is generally different from the product of the transforms of the factors,

$$\mathcal{L}(fg) \neq \mathcal{L}(f)\mathcal{L}(g) \quad ingeneral.$$

To see this take $f = e^t$ and $g = 1$. Then $fg = e^t$, $\mathcal{L}(fg) = 1/(s - 1)$, but $\mathcal{L}(f) = 1/(s - 1)$ and $\mathcal{L}(1) = 1/s$ gives $\mathcal{L}(f)\mathcal{L}(g) = \frac{1}{s^2 - s}$.

According to the next theorem, the correct answer is that $\mathcal{L}(f)\mathcal{L}(g)$ is the transform of the convolution of f and g, denoted by the standard notation $f * g$ and defined by the integral

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau \tag{15}$$

Theorem 9.1. *If two functions f and g satisfy the assumption in the existence theorem in Sec. 6.1, so that their transforms F and G exist, the product $H = FG$ is the transform of h given by* (15).

Proof. We prove the Convolution Theorem. We can denote them as we want, for instance, by τ and p, and write

$$F(s) = \int_0^\infty e^{-s\tau} f(\tau) d\tau, \quad and \; G(s) = \int_0^\infty e^{-sp} f(p) dp.$$

We now set $t = p + \tau$, where τ is at first constant. Then $p = t - \tau$, and t varies from τ to ∞. Thus

$$G(s) = \int_0^\infty e^{-s(t-\tau)} g((t-\tau))dt = e^{s\tau} \int_0^\infty e^{-s(t)} g((t-\tau))dt,$$

τ in F and t in G vary independently. Hence we can insert the G-integral into the F-integral. Cancellation of $e^{-s\tau}$ and $e^{s\tau}$ then gives

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau) e^{s\tau} \int_\tau^\infty e^{-st} g(t-\tau) dt d\tau = \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t-\tau) dt d\tau$$

Here we integrate for fixed τ over t from τ to ∞ and then over from 0 to ∞. Under the assumption on f and g the order of integration can be reversed. We then integrate first over τ from 0 to t and then over t from 0 to ∞, that is,

$$F(s)G(s) = \int_0^\infty e^{-s\tau} \int_\tau^\infty f(\tau) g(t-\tau) dt d\tau = \int_\tau^\infty e^{-st} h(t) dt = \mathcal{L}(h) = H(s).$$

□

Example 9.2.

Let $H(s) = 1/[(s-a)s]$. Find $h(t)$.

Convolution $1/[(s-a)s]$ has the inverse $f(t) = e^{at}$, and $1/s$ has the inverse $g(t) = 1$. With $f(\tau) = e^{a\tau}$ and $g(t-\tau) = 1$ we thus obtain from (15) the answer

$$h(t) = e^{at} * 1 = \int_0^t e^{a\tau} 1 d\tau = \frac{1}{a}(e^{at} - 1).$$

To check, calculate

$$H(s) = \mathcal{L}(h)(s) = \frac{1}{a}(\frac{1}{s-a} - \frac{1}{s}) = \frac{1}{a}\frac{a}{s^2 - as} = \frac{1}{s-a}\frac{1}{s} = \mathcal{L}(e^{at})\mathcal{L}(1).$$

<u>**UNIT-IV**</u>

Fourier Analysis

The aim of the study of Fourier is to express or approximate by sum of simpler trigonometric functions *i.e* the process of decomposing a function into simpler pieces.

Periodic Functions and Fourier Series

A function *f: R→R* is said to be periodic if there exists some positive real number *p* such that *f(x + p) = f(x)* for all real numbers *x*. The smallest real number *p* with this property is the period of the periodic function *f*.

Note: Linear combination of two periodic functions with same period is also a periodic function of that period.

Examples: Some familiar examples of periodic functions are *sine* functions, *cosine* functions, all constant functions etc.

Examples of functions that are not periodic are real exponential functions, non-constant polynomial functions with real coefficients etc.

Trigonometric Series

Trigonometric series are of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{n} X_i \left(a_n \cos nx + b_n \sin mx\right) ....(1)$$

are needed in the treatment of many physical problems that lead to partial differential equation, for instance, in the theory of sound, heat conduction, electromagnetic waves and mechanical vibrations.

An important advantage of the series (1) is that it can represent very general function with many discontinuities - like the discontinuous "impulse" function of electrical engineering - where as power series derivatives of all orders.

We begin our treatment with some classical calculations that were first performed by Euler. Our point of view is that the function $f(x)$ in (1) is defined on the closed interval $-\pi \le x \le \pi$, and we must find the co-efficients a_n and b_n in the series expansion. It is convenient to assume, temporarily, that the series is uniformly convergent, because this implies that the series can be integrated term by term from $-\pi$ to π.

Since $\int_{-\pi}^{\pi} \cos nx dx = 0$ and $\int_{-\pi}^{\pi} \cos nx dx = 0$ ……(2)

for $n = 1, 2, ...$, the term-by-term integration yields $\int_{-\pi}^{\pi} \cos nx dx = a_0 \pi$

so $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$.......(3)

It is worth nothing here that formula (3) shows that the constant term $\frac{1}{2}a_0$ in (1) is simply the average value of $f(x)$ over this interval. The co-efficient an is found in a similar way. Thus, if we multiply (1) by $\cos nx$ the result is

$$f(x)\cos nx = \frac{1}{2}a_0\cos nx + ... + a_n\cos^2 nx + ...(4)$$

where the terms not written contain products of the form $\sin mx\cos nx$ or of the from $\cos mx\cos nx$ with $m \neq n$. At this point it is necessary to recall the trigonometric identities

$$\sin mx\cos nx = \frac{1}{2}\left[\sin\left(m+n\right)x + \sin\left(m-n\right)x\right],$$

$$\cos mx\cos nx = \frac{1}{2}\left[\cos\left(m+n\right)x + \cos\left(m-n\right)x\right],$$

$$\sin mx\cos nx = \frac{1}{2}\left[\cos\left(m-n\right)x + \cos\left(m+n\right)x\right].$$

It is now easy to verify that for integral values of m and $n \leq 1$ we have $\int_{-\pi}^{\pi}\sin mx\cos nxdx = 0..........(5)$

and $\int_{-\pi}^{\pi}\cos mx\cos nxdx = 0, m \neq n..........(6).$

These fiats enable us to integrate (4) term by term and obtain

$$\int_{-\pi}^{\pi}\cos f(x)\cos nxdx = a_n\int_{-\pi}^{\pi}\cos^2 nxdx = a_n\pi,$$

so $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos f(x)\cos nxdx.........(7).$

By (3), formula (7) is also valid for $n = 0$; this the reason for writing the constant term in (1) as $\frac{1}{2}a_0$ rather than a_0. We get the corresponding formula for b_n by essentially the same procedure- w multiply (1) through by $\sin nx$, integrate term by term, and use the additional fact that $\int_{-\pi}^{\pi}\sin mx\sin nxdx = 0, m \neq n..........(8).$

This yields $\int_{-\pi}^{\pi} f(x)\sin nxdx = b_n\int_{-\pi}^{\pi}\sin^2 nxdx = b_n\pi,$

so $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx.........(9).$

These calculations show that if the series (1) is uniformly convergent, then the co-efficients a_n and b_n can be obtained from the sum $f(x)$ by means of above formulas. However, this situation is too restricted to be of much practical value, because how do we know whether a convergent trigonometric series?

We don't - and for this reason it is better to set aside the idea of fining the co-efficients a_n and b_n in an expansion (1) that may or not exists, and instead use formulas (7) and (9) to define certain numbers a_n and b_n that are then used to construct the trigonometric series (1). When this is done, these a_n and b_n are called the Fourier co-effiecients of the function $f(x)$, and the series (1) is called the Fourier of $f(x)$.

Just as being a Fourier series does not imply convergence, for a trigonometric series does not imply that it is a Fourier series. For example, it is known that

$$\sum_{n=1}^{\alpha} \frac{\sin mx}{\log(1+n)} ....(10)$$

Converges for ever values of *x*, and yet this series is known not to be a Fourier series.

This means that the co-efficients in (10) cannot be obtained by applying formulas (7) and (9) to any integral function $f(x)$, not even if we make the obvious choice and take the series.

A function $f(x)$ is said to be periodic if $f(x+p)=f(x)$ for all values of *x*, where *p* is a positive constant. Any positive number *p* with this property is called a period of $f(x)$.

The Fourier series of the function $f(x)=x, -\pi \le x \le \pi$. is

$$2(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} ....).........(11)$$

$\sin x$ in (11) has periods $2\pi, 4\pi, ...,$ and $\sin 2x$ has periods $\pi, 2\pi, ...$.

It is easy to see that each term of the series (11) has period 2π - in fait, 2π is the smallest period common to all the terms - so the sum also has period 2π.

Convergency:

We being by pointing out that each term of the series

$$f(x) = \frac{1}{2}a_0 + \sum_{1}^{\alpha}\left(a_n \cos nx + b_n \sin nx\right).......(1)$$

has period 2π, and therefore, if the function $f(x)$ is to be represented by the sum, $f(x)$ must also have period 2π. Whenever we consider a series like (1), we shall assume that $f(x)$ is initially given on the basic interval $-\pi \le x \le \pi$ or $-\pi \le x \le \pi$, and that for other values of *x*, $f(x)$ is defined by the periodicity condition

$$f(x+2\pi) = f(x)......(2).$$

In particular, (2) requires that we must always have $f(\pi) = f(-\pi)$. Accordingly, the complete function we consider is the so-called “periodic extension” of the originally given part of the successive intervals of length 2π that lie to the right and left of the basic interval.

Dirichlet's conditions: A function $f(x)$ is said to have satisfied Dirichlet's conditions in the interval (-*L*,*L*), Provided $f(x)$ is periodic, piecewise continuous, and has a finite number of relative maxima and minima in (-*L*,*L*).

Note: The phrase simple distribution (or often jump discontinuity) is used to finite jump at a point to describe the situation where a function has a finite jump at a point $x = x_0$. This means that $f(x)$ approaches finite but different limits from the left side x_0 and from the right side.

If a bounded function $f(x)$ has only a finite number of discontinuities and only a finite. Numbers of maxima and minima, then all the discontinuities are simple. The function defined by $f(x) = \sin\frac{1}{x}(x \neq 0), f(0) = 0$ has infinitely many maxima near $x = 0$ and the discontinuity at $x = 0$ not simple. The function defined by $g(x) = x\sin\frac{1}{x}(x \neq 0)\, g(0) = 0$ and $h(x) = x^2 \sin\frac{1}{x}(x \neq 0), h(0) = 0$.

Also have infinitely many maxima near $x = 0$, but both are continuous at $x = 0$ where as only $h(x)$ is differentiable at this point.

The general situation is as follows:

The continuity of a function is not sufficient for the convergence of its Fourier series to the function, and neither it is necessary. That is it is quite possible for a discontinuous function to be represented everywhere by its Fourier series, provided it is relatively well- behaved between the points of discontinuity. In Dirichlet's condition, the discontinuities are simply and the graph consists of a finite number of increasing or decreasing continuous pieces.

Theorem: If a function f is bounded and integrable in $[0,a], a >$ and monotone in $[0,\delta]$, $0 < \delta < a$, then

$$\lim_{n\to\alpha}\int_0^a f\frac{\sin nx}{x}dx = f(0^+)\int_0^\alpha \frac{\sin x}{x}dx.$$

Note: Integrals of the following two forms are called Dirichlet's integrals.

$$\int_0^a f\frac{\sin nx}{\sin x}dx, \int_0^\alpha f\frac{\sin x}{x}dx$$

Theorem: If a function f is bounded periodic with period 2π and integrable on $[-\pi,\pi]$ and piecewise monotonic on $[-\pi,\pi]$, then

$$\frac{1}{2}a_0 + \sum_{n=1}^{\alpha}(a_n \cos nx + b_n \sin nx)$$

$$= \begin{cases} \frac{1}{2}\left\{ f(x-0) + \frac{1}{2}f(x+0)\right\}, & \text{for } -\pi < x < \pi \\ \frac{1}{2}\left\{ f(\pi-0) + f(-\pi+0)\right\}, & \text{for } x = \pm\pi, \end{cases}$$

where a_n, b_n are Fourier co-efficents series of *f*, and *x* a point of $[-\pi, \pi]$.

The *mth* partial sum at point ξ,

Proof: Let $\frac{1}{2}a_0 + \sum_{n=1}^{m}(a_n \cos nx + b_n \sin nx)$ be the Fourier series of *f*, and x a point of $[-\pi, \pi]$.

The *mth* partial sum at the point ξ,

$$\frac{1}{2}a_0 + \sum_{n=1}^{m}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{n=1}^{m}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)(\cos nt \cos nx + \sin nt \sin nx)dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left[1 + 2\sum_{n=1}^{m}\cos nlt - x\right]dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\left[1 + 2\sum_{n=1}^{m}\cos nt\right]dt$$

$$\left(\because \text{ For a periodic function of period } 2\pi, \int_{\alpha}^{\beta} fdx = \int_{\alpha+2\pi}^{\beta+2\pi} fdx\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} f(x+t)\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt + \frac{1}{2\pi}\int_{0}^{\pi} f(x+t)\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}dt$$

$$= \frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}} f(x-2t')\frac{\sin(2m+1)t'}{\sin t'}dt' + \frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}} f(2t'+x)\frac{\sin(2m+1)t'}{\sin t'}dt'$$

Proceeding to limits when $m \to \alpha$

$$\frac{1}{2}a_0\sum_{n=1}^{\alpha}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\pi f(x-0) + \frac{1}{2}\pi f(x+0)\right]$$

$$= \frac{f(x-0) + f(x+0)}{2} x \in [-\pi, \pi]$$

Thus the Fourier series of a (periodic) function f which is bounded, integrable and piecewise monotonic on $[-\pi, \pi]$, converges to $\frac{1}{2}[f(x-0) + f(x+0)]$ at a point *x*, $-\pi < x < \pi$, and (using periodicity of *f*) to $\frac{1}{2}[f(x-0) + f(x+0)]$ at the ends, $\pm\pi$.

Half –range series:

(A) Cosine series: Let $f(x)$ satisfy Dirichlet's condition in $0 \le x \le \pi$, then

$$\frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx$ is called Fourier cosine series corresponding to $f(x)$ in the interval. The series is equal to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at every x-in $0 < x < \pi$ where $f(x+0)$ and $f(x-0)$ exists and is equal to $f(0+0)$ at $x=0$ and equal to $f(\pi-0)$ at $x=\pi$, provided both $f(0+0)$ and $f(\pi-0)$ exists.

(B) Sine series: If $f(x)$ satisfies Dirichlet's conditions in $0 < x < \pi$, then $\sum_{n=1}^{\alpha} b_n \sin nx$ where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nxdx$ represents $f(x)$ in Fourier sine series in the interval. The series is equal to $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point x in $0 < x < \pi$ when $f(x+0)$ and $f(x-0)$ exists and when $x=0$ and $x=\pi$ the sum is zero.

The Fourier series in other intervals:

(A) In $[0, 2\pi]$: If $f(x)$ satisfies Dirichlet's conditions in $0 \le x \le 2\pi$, then the sum of the series $\frac{1}{2}a_0 + \sum_{n=1}^{\alpha}(a_n \cos nx + b_n \sin mx)$, where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx \\ \text{and } b_n &= \frac{1}{\pi}\int_0^{d\pi} f(x)\sin mxdx \end{aligned}\right\} n \le 1.$$

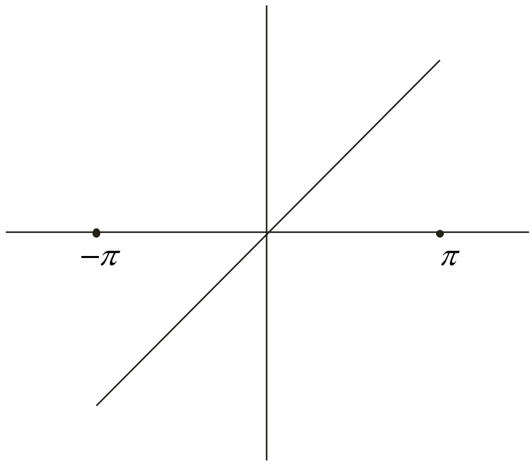
is $\frac{f(x_0+0)+f(x_0-0)}{2}$ at any point x_0 in $0 \le x \le 2\pi$ and is $\frac{f(x_0+0)+f(x_0-0)}{2}$ at $x=0$ 2π and is periodic with period 2π.

(Follow from Mallik & Arrora)

Ex.1: If the trigonometric series $\frac{a_0}{2} + \sum_{n=1}^{\alpha}(a_n \cos nx + b_n \sin nx)$ is uniformly convergent on $[-\pi, \pi]$ and if $f(x)$ be its sum then prove that it is the Fourier series of $f(x)$ in $[-\pi, \pi]$.

Ex.2: Expand $f(x)=x$ in Fourier series in the interval $-\pi \le x \le \pi$. What soes the series represent for other values of *x*? what is sum of the series for $x=\pm\pi$ and $x=0$?

Sol: Obviously the function $f(x)=x$ is bounded and integrable on $-\pi \le x \le \pi$, since it is continuous there. Further $f^{/}(x)=1>0$ indicates that $f(x)$ is monotonic increasing the entire interval. Thus $f(x)$ satisfies Dirichlet's conditions on $[-\pi,\pi]$.



Hence the Fourier series Corresponding to $f(x)=x$

is $\frac{a_0}{2}+\sum_{n=1}^{\alpha}(a_n \cos nx + b_n \sin nx)$

where $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx = 0, a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} x dx = 0.$

Since $x\cos nx$ and x are odd functions, and

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin mx dx=\frac{2}{\pi}\int_{0}^{\pi} x\sin mx dx$$

Since $x\sin mx$ is even. Thus

$$b_n=\frac{2}{\pi}\left[-x\frac{\cos x}{n}\right]_0^{\pi}+\frac{2}{\pi}\int_0^{\pi}\frac{\cos nx}{n}dx$$

$$=-\frac{2}{n}\cos nm=\begin{cases}-\frac{2}{n}, & n \text{ is even}\\ \frac{2}{n}, & n \text{ is odd}\end{cases}$$

Hence $f(x)=x$ generates Fourier series in the form

$$\frac{a_0}{2}+\sum_{n=1}^{\alpha}(a_n \cos nx + b_n \sin nx)=\sum_{n=1}^{\alpha} b_n \sin nx$$

$$=b_1 \sin nx + b_2 \sin 2x + b_3 \sin 3x + ...$$

$$=2\left\{\frac{\sin nx}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}.........\right\}.....(1)$$

2nd part: For other values of x it converges to the periodic extension.

3rd part: At $x\pm\pi,$ the sum of the series (1) is $\frac{f(-\pi+0)+f(-\pi+0)}{2}=\frac{-\pi+\pi}{2}=0$.

At $x=0$, the sum of the series (1) is $f(0)=0$, (since $f(x)$ is continuous at $x=0$).

Ex.3: Expand Fourier series $x+x^2$ on $-\pi \le x \le \pi$ and deduce that

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^3}+.......$$

Sol: Let $f(x)=x+x^2$ on $-\pi \le x \le \pi$. We may define $f(x)$ at $x=\pm\pi$ arbitrary. Let us take $f(x)=x+x^2$ on $-\pi \le x \le \pi$ and $f(\pi)=f(-\pi)$.

Now f is bounded and integrable on $[-\pi,\pi]$. Further $f^{/}(x)<1+2x$, so that $f^{/}(x)<0$ for $x<-\frac{1}{2}$ and $f^{/}(x)<0$ for $x<-\frac{1}{2}$. Thus $f(x)$ is monotonic decreasing on $-\pi\le x\le -\frac{1}{2}$ and monotonic increasing on $-\frac{1}{2}\le x\le \pi$ whereby f is piecewise monotonic on $-\pi\le x\le \pi$. Hence $f(x)$ satisfies Dirichlet's conditions on $[-\pi,\pi]$. Now

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx dx=\frac{4}{n^2}\cos n\pi=\begin{cases}\frac{4}{n^2},\ n\text{ even}\\ -\frac{4}{n^2},\ n\text{ odd.}\end{cases}$$

Similarly $b_n=-\frac{2}{n}$ where n is even and $b_n=\frac{2}{n}$ when n is odd. Thus

$$x+x^2 \sim \frac{a_0}{2}+\sum_{n=1}^{\alpha}(a_n\cos nx+b_n\sin nx)$$

$$=\frac{\pi^2}{3}-4\left\{\frac{\cos nx}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}.........\right\}+2\left\{\frac{\sin nx}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}.........\right\}$$

And $x+x^2$ is a continuous function; hence on $-\pi\le x\le\pi$,

$$x+x^2=\frac{\pi^2}{3}-4\left\{\frac{\cos nx}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}.........\right\}+2\left\{\frac{\sin nx}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}.........\right\}$$

At $x=\pm\pi$, the sum of the series

$$=\frac{1}{2}\{f(-\pi+0)+f(-\pi+0)\}=\frac{1}{2}(-\pi+\pi^2+\pi+\pi^2)=\pi^2$$

$$\therefore \pi^2=\frac{\pi^2}{3}-4\left(-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}.......\right)$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}....=\frac{\pi^2}{6}.$$

Ex.4: Expand in a series of sines and cosines of multiple of x, the function.

$$f(x)=\begin{cases}x-\pi,-\pi<x<0\\ x-\pi,0<x<\pi.\end{cases}$$

Sol: See that $f(x)$ is not defined at $x=0$, $\pi,-\pi$ where it can be defined in any manner. Let us take $f(x)=-\pi$ at $x=0$ and $f(x)=0$ at $x=\pm\pi$. Obviously the function $f(x)$ is bounded in $-\pi\le x\le\pi$. Obviously the function $f(x)$ is bounded in $-\pi\le x\le\pi$, -2π and π being the bounds. Moreover, $f(x)$ is monotone increasing on $-\pi\le x\le 0$ and $f(x)$ is monotone decreasing on $0<x<\pi$ whereby

$f(x)$ is piecewise monotone on $-\pi \le x \le \pi$. Hence $f(x)$ satisfies Dirichlet's conditions on $[-\pi, \pi]$.

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(x-\pi)dx + \int_{0}^{\pi}(\pi - x)dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2} - \pi x\right]_{-\pi}^{0} + \left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\left(\frac{x^2}{2} - \pi^2\right) + \left(\pi^2 - \frac{x^2}{2}\right)\right\} = -\pi$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos xdx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(x-\pi)\cos nxdx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nxdx\right\}$$

$$= \frac{1}{\pi}\left\{\left[(x-\pi)\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \left[(x-\pi)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{-\pi}^{0}\right\}$$

$$= \frac{2}{n^2\pi}(1 - \cos n\pi) = \frac{2\left\{1-(-1)^n\right\}}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(x-\pi)\sin mxdx + \int_{-\pi}^{0}(x-\pi)\sin nxdx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-(x-\pi)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \left[-(x-\pi)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0}\right\}$$

$$= \frac{2(1-\cos n\pi)}{n} = \frac{2\left\{1-(-1)^n\right\}}{n}$$

The Fourier series corresponding to $f(x)$ on $-\pi \le x \le \pi$ is then

$$-\frac{\pi}{2} + \sum_{n=1}^{\alpha}\frac{2\left\{1-(-1)^n\right\}}{n^2}\cos nx + \sum_{n=1}^{\alpha}\frac{2\left\{1-(-1)^n\right\}}{n}\sin nx.$$

$$= -\frac{\pi}{2} + 4\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\ldots\right\} + 4\left\{\frac{\sin x}{1} + \frac{\sin 3x}{3} + \ldots\ldots\right\}\ldots..(1)$$

At $x = \pm\pi$, the sum of the series (1) is $\frac{f(-\pi+0) + f(\pi-0)}{2} = -\pi$. At $x = 0$, the sum of the series (1) is $\frac{f(0+0) + f(0-0)}{2} = \frac{\pi - \pi}{2} = 0$.

Ex.5: Represent $f(x)$ where $f(x)=\cos kx$ on $-\pi \le x \le \pi$ (K not being an integer) in Fourier series.

Deduce that

(i) $\pi \cot k\pi = \frac{1}{k} + 2k\sum_{n=1}^{\alpha} \frac{1}{k^2-n^2}$

(ii) $\frac{\pi}{\sin k\pi} = \sum_{n=0}^{\alpha} (-1)^n \left\{ \frac{1}{n+K} + \frac{1}{n+1-K} \right\}.$

Sol: Obviously the function $f(x)=\cos Kx$ is bounded and integrable and piecewise monotonic in the interval $-\pi \le x \le \pi$. Therefore monotone in the interval $-\pi \le x \le \pi$. Therefore $f(x)$ satisfies Dirichlet's conditions on $[-\pi, \pi]$.

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos Kx dx = \frac{2\sin K\pi}{K\pi}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos Kx \cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \cos Kx \cos nx dx \left[\text{as } \cos Kx \cos nx \text{ ia an even function}\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \{\cos(k+n)x + \cos(K-n)x\} dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(K+n)x}{K+n} + \frac{\sin(K-n)x}{K-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(K+n)x}{K+n} + \frac{\sin(K-n)x}{K-n}\right]$$

$$= \frac{1}{\pi}(-1)^n \sin K\pi.\left(\frac{2k}{k^2-n^2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos Kx \sin x dx = 0$$

$\left[\text{as } \cos Kx \sin nx \text{ ia an odd function}\right]$

The Fourier series corresponding to $f(x)$ on $-\pi \le x \le \pi$ is then

$$\frac{\sin K\pi}{K_n} + \sum_{n=1}^{\alpha} \frac{2K(-1)^n}{\pi(k^2-n^2)} \sin nK\pi \cos nx......(1)$$

(i) At $x = \pm\pi,$ the sum of the series (1) is $\frac{f(-\pi+0)+f(\pi-0)}{2}$

$= \cos K\pi.$

$$\therefore \cos K\pi = \frac{\sin K\pi}{K\pi} + \sum_{n=1}^{\alpha} \frac{2K(-1)}{\pi(K^2-n^2)} \sin K\pi \cos n\pi$$

or, $\pi \cot K\pi = \frac{1}{K} + 2K\sum_{n=1}^{\alpha} \frac{1}{K^2-n^2}$

(ii) At $x=0$, the sum of the series is 1,

$$\therefore 1 = \frac{\sin K\pi}{K\pi} + \sum_{n=1}^{\alpha} \frac{2K(-1)^n}{\pi(K^2 - n^2)} \sin K\pi$$

$$\frac{\sin K\pi}{K\pi} = \frac{1}{K} + \sum_{n=1}^{\alpha} \frac{2K(-1)^n}{(K^2 - n^2)} = \frac{1}{K} + \sum_{n=1}^{\alpha} (-1)^n \left\{ \frac{1}{K-n} + \frac{1}{K-n} \right\}$$

$$= \frac{1}{K} + \sum_{n=1}^{\alpha} (-1)^n \left\{ -\frac{1}{K-n} + \frac{1}{K-n} \right\}$$

$$= \frac{1}{K} - \left(-\frac{1}{1-K} + \frac{1}{1+K} \right) + \left(-\frac{1}{2-K} + \frac{1}{2+K} \right) - \left(-\frac{1}{3-K} + \frac{1}{3+K} \right) + \dots\dots\dots$$

$$= \left(\frac{1}{K} + \frac{1}{1-K} \right) - \left(\frac{1}{1+K} + \frac{1}{2-K} \right) + \dots\dots\dots\dots$$

$$= \sum_{n=0}^{\alpha} (-1)^n \left\{ \frac{1}{1+K} + \frac{1}{n+1+K} \right\}$$

Note: A half range Fourier series is a Fourier series defined on an interval [*0,L*] instead of the more common [-*L,L*]. The function f(x), x ϵ [*0,L*] can be extended to [-*L,0*] for an even or odd function *f(x).* Hence, the even extension generates the *cosine* half range expansion and the odd extension generates the *sine* half range expansion.

ASSIGNMENT

Q 1: Let

$$f(x) = x, 0 \le x \le \frac{\pi}{2},$$
$$= \pi - x, \frac{\pi}{2} \le x \le \pi,$$
$$= -f(-x), -\pi \le x \le 0.$$

verify that f satisfies Dirichlet's condition on $[-\pi, \pi]$. Obtain the Fourier series for f in $[-\pi, \pi]$.

Q 2: Show that Fourier series corresponding to x^2 on $-\pi \le x \le \pi$ is $\frac{\pi^2}{3} + 4\sum_{n=1}^{\alpha}(-1)\frac{\cos nx}{n^2}$ and hence deduce that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\dots = \frac{\pi^2}{6}, 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \dots\dots = \frac{\pi^2}{12}; 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots = \frac{\pi^2}{8}$

Q 3: If

$$f(x) = -x \text{ for } -\pi \le x \le \pi$$
$$= 0 \quad \text{for } 0 \le x \le \pi$$

Then show that Fourier series corresponding to $f(x)$ on $-\pi \le x \le \pi$ is

$$\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\alpha}\frac{\cos(2n-1)x}{(2n-1)^2} + \sum_{n=1}^{\alpha}(-1)^2\frac{\sin nx}{n}$$

Q 4: prove that the even function $f(x) = |x|$ on $-\pi \le x \le \pi$ has a cosine in Fourier's form as

$|x| \square \frac{\pi}{2} - \frac{4}{\pi}\left\{\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right\}$ Apply Dirichlet's conditions of convergences to show that he series converges to $|x|$ throughout $-\pi \le x \le \pi$. Also show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8}$.

Q 5: Show that e^{ax} on $-\pi \le x \le \pi$ represents

$$e^{ax}\frac{e^{a\pi} - e^{-a\pi}}{\pi}\left\{\frac{1}{2a} + \sum_{n=1}^{\alpha}\frac{(-1)}{n^2 + a^2}(a\cos nx - n\sin nx)\right\}.$$

Q 6: Show that on $-\pi \le x \le \pi$, $\frac{\pi}{2\sinh\pi}e^{x} = \frac{1}{2} + \sum_{n=1}^{\alpha}(-1)^n\frac{(\cos nx - n\sin nx)}{1+n^2}$

Q 7: Find a Fourier series representing $f(x)$ on $-\pi \le x \le \pi$ when

$$f(x)0, \quad -\pi \le x \le \pi$$
$$= \frac{1}{4}\pi x, \quad 0 < x < \pi.$$

and deduce that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8}$.

Q 8: Find the Fourier series of the periodic function f with period 2π defined as follows:

$$f(x) = 0, \quad \text{for } -\pi \le x \le \pi$$
$$= x, \quad \text{for } 0 < x < \pi.$$

what is the sum of the series at $x = 5\pi$? Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$.

Q 9: What do you mean by the Fourier service of a function f which is defined, bounded, and integrable on $[-\pi, \pi]$. Find the Fourier cosine which represents the periodic function $f(x) = x$ in $0 < x < \pi$.

Q 10: Expand the function $|x|$ in as Fourier series on $[-1, 1]$.

Q 11: Expand $f(x) = \begin{cases} \sinh \pi x, & 0 < x < \frac{1}{2} \\ 0, & \frac{1}{2} < x < 1. \end{cases}$

In a series of the form $\sum_{n=1}^{\alpha} a_n \sin n\pi x$ and deduce that $\sum_{n=1}^{\alpha} \frac{1}{n^2+(n-1)^2} = \frac{\pi}{2}\tanh\frac{\pi}{2}$.

Q 12: Expand $f(x)$ in Fourier sine series on $0 \le x \le \pi$.
where

$$f(x) = \frac{\pi x}{4}, \quad 0 \le x < \frac{\pi}{2}$$
$$= \frac{\pi}{2}(\pi - x), \quad \frac{\pi}{2} < x < \pi.$$

Q 13: Find the Fourier expansion for $f(x)$ which is periodic with period 2π and which on $0 \le x \le 2\pi$ is given by $f(x) = x^2$.
Find the sum of the series at $x = 4\pi$ and hence show that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots = \frac{\pi^2}{6}$.

Q 14: If a_n and b_n are the Fourier co-eeficients of the function $f(x)$ defined $\frac{a_0}{2}+\sum_{K=1}^{n}(a_K \cos Kx + b_K \sin Kx)$ by the interval $-\pi \le x \le \pi$ show that $= \int_{-\pi}^{\pi} \frac{f(x+t)}{2\pi} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}} dt.$