Subject: B. Tech. PHYSICS – I (3 – 1 – 0)

Waves and Oscillations

Periodic & Oscillatory Motion:-

The motion in which repeats after a regular interval of time is called periodic motion.

1. The periodic motion in which there is existence of a restoring force and the body moves along the same path to and fro about a definite point called equilibrium position/mean position, is called oscillatory motion.
2. In all type of oscillatory motion one thing is common i.e each body (performing oscillatory motion) is subjected to a restoring force that increases with increase in displacement from mean position.
3. **Types of oscillatory motion:-**

   It is of two types such as linear oscillation and circular oscillation.

   Example of linear oscillation:-

   1. Oscillation of mass spring system.
   2. Oscillation of fluid column in a U-tube.
   3. Oscillation of floating cylinder.
   4. Oscillation of body dropped in a tunnel along earth diameter.
   5. Oscillation of strings of musical instruments.

   Example of circular oscillation:-

   1. Oscillation of simple pendulum.
   2. Oscillation of solid sphere in a cylinder (If solid sphere rolls without slipping).
   3. Oscillation of a circular ring suspended on a nail.

4. Oscillation of balance wheel of a clock.
5. Rotation of the earth around the sun.

<u>Oscillatory system</u>:-

1. The system in which the object exhibit to & fro motion about the equilibrium position with a restoring force is called oscillatory system.
2. Oscillatory system is of two types such as mechanical and non- mechanical system.
3. **<u>Mechanical oscillatory system</u>:-**
    - In this type of system body itself changes its position.
    - For mechanical oscillation two things are specially responsible i.e Inertia & Restoring force.
    - E.g oscillation of mass spring system, oscillation of fluid-column in a U-tube, oscillation of simple pendulum, rotation of earth around the sun, oscillation of body dropped in a tunnel along earth diameter, oscillation of floating cylinder, oscillation of a circular ring suspended on a nail, oscillation of atoms and ions of solids, vibration of swings…etc.

4.<u>Non-mechanical oscillatory system</u>:-

In this type of system, body itself doesn't change its position but its physical property varies periodically.

e.g:-The electric current in an oscillatory circuit, the lamp of a body which is heated and cooled periodically, the pressure in a gas through a medium in which sound propagates, the electric and magnetic waves propagates undergoes oscillatory change.

<u>Simple Harmonic Motion</u>:-

It is the simplest type of oscillatory motion.

A particle is said to be execute simple harmonic oscillation is the restoring force is directed towards the equilibrium position and its magnitude is directly proportional to the magnitude and displacement from the equilibrium position.

If F is the restoring force on the oscillator when its displacement from the equilibrium position is x, then

$$F \propto -x$$

Here, the negative sign implies that the direction of restoring force is opposite to that of displacement of body i.e towards equilibrium position.

$$F = -kx \ \ldots\ldots\ldots\ldots (1)$$

Where, k= proportionality constant called force constant.

$$Ma = -kx$$

$$M\frac{d2y}{dt2} = -kx$$

$$M\frac{d2y}{dt2} + kx = 0$$

$$\frac{d2y}{dt2} + \frac{k}{M}x = 0$$

$$\frac{d2y}{dt2} + \omega^2 x = 0 \ \ldots\ldots\ldots\ldots (2)$$

Where $\omega^2=\frac{k}{M}$

Here $\omega=\sqrt{\frac{k}{M}}$ is the angular frequency of the oscillation.

Equation (2) is called general differential equation of SHM.

By solving these differential equation

$x=\alpha e^{-i\omega t} + \beta e^{i\omega t}$ ……… (3)

Where α, β are two constants which can be determined from the initial condition of a physical system.

Appling de-Moiver's theorem

$x= \alpha(\cos\omega t+i\sin\omega t) + \beta(\cos\omega t-i\sin\omega t)$

$x= (\alpha+ \beta) \cos\omega t+ (\alpha - \beta) \sin\omega t$

$x= C\cos\omega t+D\sin\omega t$ ………(4)

Where $C = \alpha + \beta$

& $D= \alpha - \beta$

Let assume,

$C=A\sin\theta$

$D=A\cos\theta$

Putting these value in equation (4)

$x=A\sin\theta\cos\omega t +A\cos\theta\sin\omega t$

$x=A (\sin\theta\cos\omega t + \cos\theta \sin\omega t)$

$x=A \sin(\omega t + \theta)$ ……….. (5)

Where $A=\sqrt{(C^2+D^2)}$ & $\theta = \tan^{-1}(\frac{C}{D})$

Similarly, the solution of differential equation can be given as

x=Acos($\theta + \omega t$) ………(6)

Here A denotes amplitude of oscillatory system, ($\theta + \omega t$) is called phase and θ is called epoch/initial phase/phase constant/phase angel.

Equation (5) and (6) represents displacement of SHM.

Velocity in SHM:-

$$x\text{=Asin}(\omega t + \theta)$$

$$\frac{dx}{dt}\text{=A}\omega\cos(\omega t + \theta)$$

$$\text{v=A}\omega\cos(\omega t + \theta) \text{ ………… (7)}$$

The minimum value of v is 0(as minimum value of Asin($\theta + \omega t$)=0 & maximum value is Aω. The maximum value of v is called velocity amplitude.

Acceleration in SHM:-

$$\text{a= -A}\omega^2\sin(\omega t + \theta) \text{ …………. (8)}$$

The minimum value of ‘a’ is 0 & maximum value is Aω^2. The maximum valueof ‘a’ is called acceleration amplitude.

Also, a= ω^2x (from equation (5))

a ∝ –y

It is also the condition for SHM.

Time period in SHM:-

The time required for one complete oscillation is called the time period (T). It is related to the angular frequency(ω) by.

$$\text{T=}\frac{2\pi}{\omega} \text{ ……………… (9)}$$

Frequency in SHM:-

The number of oscillation per time is called frequency or it is the reciprocal of time period.

$$\upsilon=\frac{1}{T}=\frac{\omega}{2\pi} \ldots\ldots\ldots\ldots\ldots(10)$$

Potential energy in SHM:-

The potential energy of oscillator at any instant of time is,

$$U=-\int_0^x F dx$$

$$=-\int_0^x (-kx)\, dx$$

$$=\frac{1}{2}\, kx^2$$

$$=\frac{1}{2}\, kA^2\sin^2(\theta+\omega t) \ldots\ldots\ldots\ldots (11)$$

(By using equation (5)).

Kinetic energy in SHM:-

The kinetic energy of oscillator at any instant of time is,

$$K=\frac{1}{2}M\left(\frac{dx}{dt}\right)^2$$

$$=\frac{1}{2}M\mathrm{v}^2$$

$$K=\frac{1}{2}MA^2\omega^2\cos^2(\theta+\omega t) \ldots\ldots\ldots (12)$$

(By using equation (7))

Both kinetic and potential energy oscillate with time when the kinetic energy is maximum, the potential energy is minimum and vice versa. Both kinetic and potential energy attain their maximum value twice in one complete oscillation.

Total energy in SHM:-

Total energy= K.E+P.E

$$=\frac{1}{2}MA^2\omega^2\cos^2(\theta+\omega t)+\frac{1}{2}kA^2\sin^2(\theta+\omega t)$$

$$=\frac{1}{2}kA^2\cos^2(\theta+\omega t)+\frac{1}{2}kA^2\sin^2(\theta+\omega t)$$

Total energy $=\frac{1}{2}kA^2$

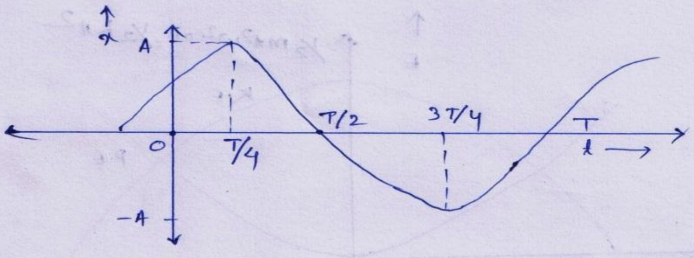
Total energy $=\frac{1}{2}MA^2\omega^2$

The total energy of an oscillatory system is constant.

<u>Graphical relation between different characteristics in SHM.</u>

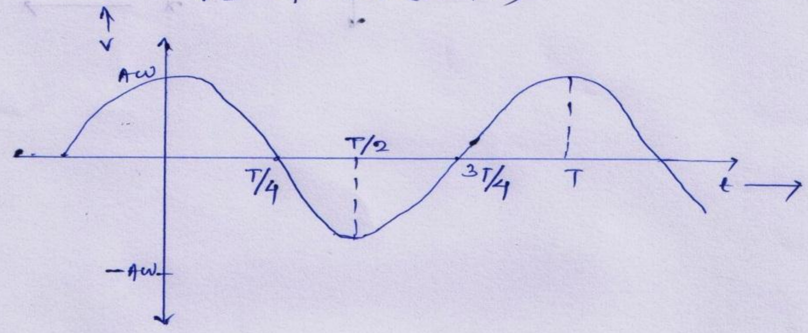
(a) Displacement ~ time.

$$x = A \sin(\omega t + \theta)$$



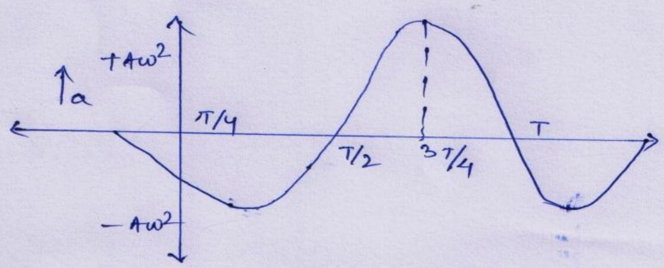
(b) Velocity ~ time

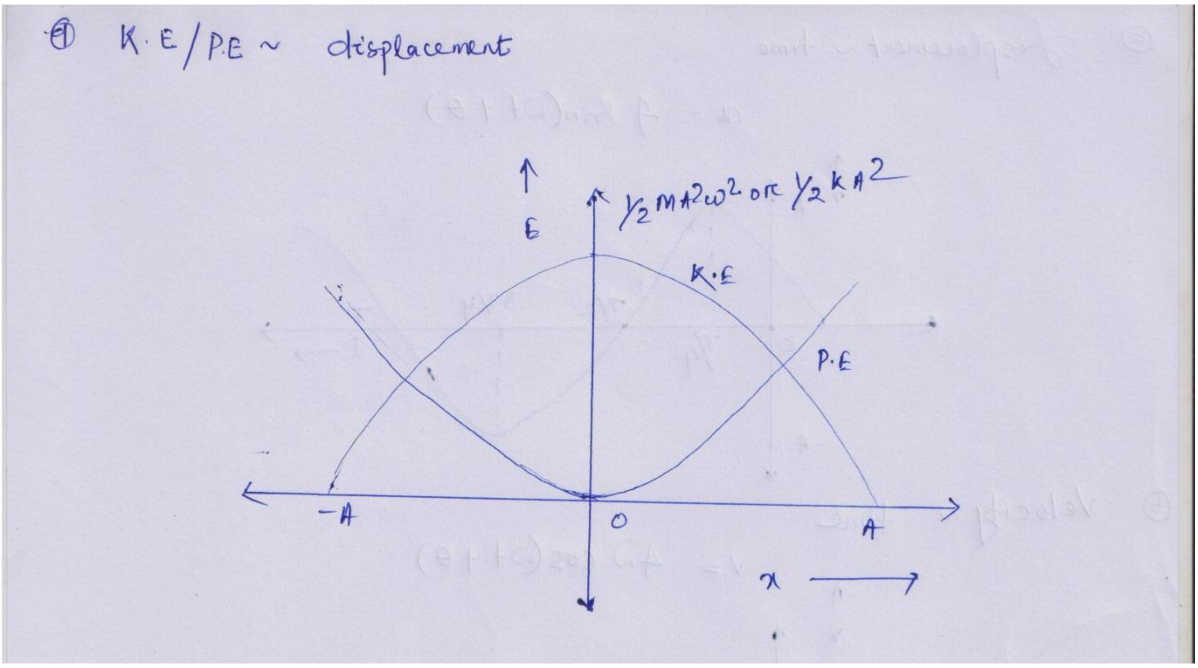
$$v = A\omega \cos(\omega t + \theta)$$



(c) Acceleration ~ time

$$a = A\omega^2 \sin(\omega t + \theta)$$



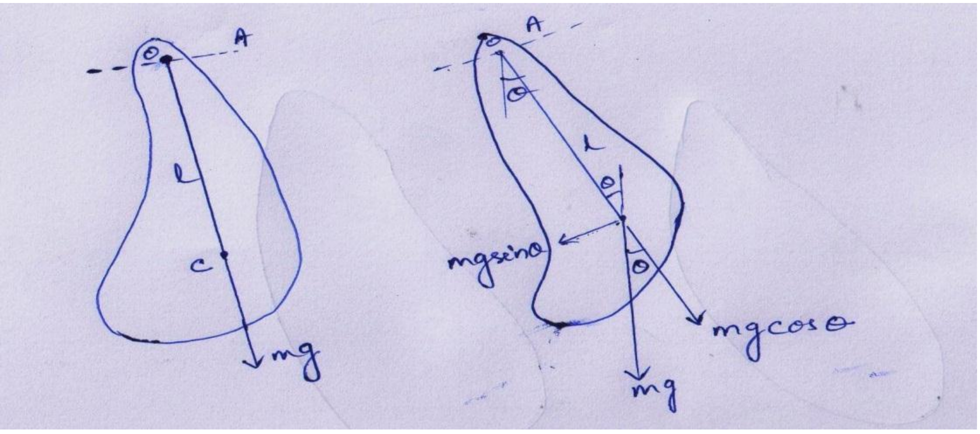


COMPOUND PENDULUM (Physical pendulum):-

Compound /physical pendulum is a rigid body of any arbitrary shape capable of rotating in a vertical plane about an axis passing through the pendulum but not through the pendulum but not through centre of gravity of pendulum.

The distance between the point of suspension the centre of gravity is called the length of length of the pendulum &denoted by

When the pendulum is displaced through a angle θ from the mean position,a restoring torque come to play which tries to bring the pendulum back to the mean position .But the oscillation continues due to the inertia of restoring force.



Here the restoring force is -mgsinθ. So the restoring torque about the point of suspension "O" is

$\tau = -mgl\sin\theta$.

If the moment of inertia of the body about "OA" is "I", the angular acceleration becomes,

$\alpha = \tau/I$

$\alpha = -mgl\sin\theta/I$……………….(1)

For very small angular displace "θ ", we assume that

Sin θ~θ.

So, $\alpha = -mgl\theta/I$.

$\alpha = -(mgl/I)\,\theta$………. (2)

Also $\alpha = d^2\theta/dt^2$

Now we can write

$d^2\theta/dt^2 + (mgl/I)\,\theta = 0$……………..(3)

$d^2\theta/dt^2 + \omega^2\theta = 0$……………..(4)

Where, $\omega^2 = mgl/I$. And eq^n(4) is the general equation of simple harmonic.

$T = 2\pi(I/mgl)^{1/2}$

$T = 2\pi(M(k^2 + L^2)/Mgl)^{1/2}$.

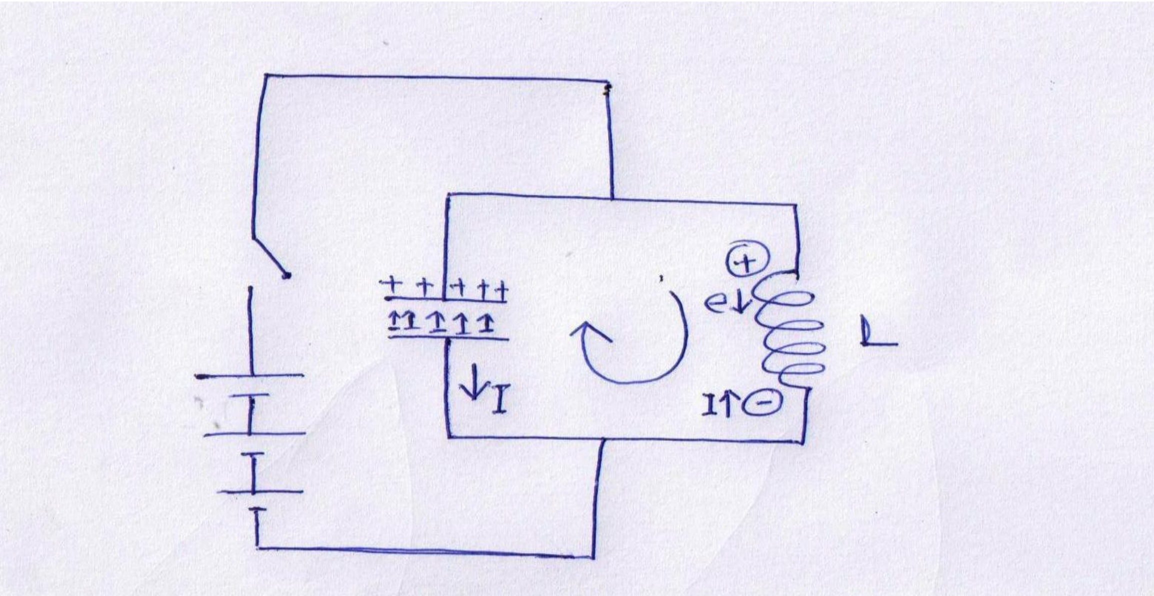
$T = 2\pi((K^2/l + l)/g)^{1/2}$……………………………(5).

Here $\frac{k2}{l} + l = L$, Called as equivalent length of pendulum..

If a line which is drawn along the line joining the point of suspension & Centre of gravity by the distance " k^2/l".we have another Point on the line called centre of Oscillation is equivalent Length of pendulum .

So,the distance between centre of suspension & centre of Oscillation is equivalent length of pendulum .If these two points are interchanged then “time period” will be constant.

L.C CIRCUIT(NON MECHANICAL OSCILLATION):-



In this region,it is combination “L” &”C” with the DC source through the key.If we Press the Key for a while then capacitor get charged & restores the charge as “+Q” and”-Q” with the potential “v=q/c” between the plates .When the switch is off the capacitor gets discharged.

As capacitor gets discharged, q also decreases. So, current at that situation is given by

I=dq/dt.

As q decreases, electric field energy (Energy stored in electric field) gradually decreases .This energy is transferred to magnetic field that appears around the inductor. At a time,all the charge on the capacitor becomes zero,the energy of capacitor is also Zero. Even though q equals to zero,the current is zero at this time.

Mathematically, Let the potential difference across the two plates of capacitor at any instance" V" is given by

V=q/c…………………(1)

In the inductor due to increases in the value of flow of current, the strength of magnetic field ultimately the magnetic lines of force cut/ link with inductor changes. So a back emfdevelops which is given by

$\varepsilon \quad = -L\frac{di}{dt}$……………(2)

Now applying KVL to this LC circuit,

+v-ε=0

$\frac{q}{c}+L\frac{di}{dt}=0$

$\frac{q}{Lc}+\frac{d2q}{d2t}=0$

$\frac{d2q}{d2t}+\frac{q}{Lc}=0$………………………..(3).

This represents the general equation of SHM,

Here there is periodic execution of energy between electric field of capacitor & magnetic field of inductor.

Here this LC oscillation act as an source of electromagnetic wave.

Here, $\omega^2 = {}^{1}/_{LC}$

$\omega = {}^{1}/_{\sqrt{LC}}$

$T = 2\pi\sqrt{LC}$

<u>Damped oscillation</u>:-

For a free oscillation the energy remains constant. Hence oscillation continues indefinitely. However in real fact, the amplitude of the oscillatory system gradually decreases due to experiences of damping force like friction and resistance of the media.

The oscillators whose amplitude, in successive oscillations goes on decreasing due to the presence of resistive forces are called damped oscillators, and oscillation called damping oscillation.

The damping force always acts in a opposite directions to that of motion of oscillatory body and velocity dependent.

$$F_{dam} \propto -v$$

$$F_{dam} = -bv$$

b= damping constant which is a positive quantity defined as damping force/velocity,

$$F_{net} = F_{res} + F_{dam}$$

$$F_{net} = -kx - bv$$

$$F_{net} = -kx - b\frac{dx}{dt}$$

$$M\frac{d2x}{dt2} + kx + b\frac{dx}{dt} = 0$$

$$\frac{d2x}{dt2} + \frac{b}{M}\frac{dx}{dt} + \frac{k}{M}x = 0$$

$$\frac{d2x}{dt2}+2\beta\frac{dx}{dt}+\omega_0^2x = 0 \text{ ………… (2)}$$

Where $\beta= \frac{b}{2M}$ is the damping co-efficient & $\omega_0=\sqrt{\frac{k}{M}}$ is called the natural frequency of oscillating body.

The above equation is second degree linear homogeneous equation.

The general solution of above equation is found out by assuming x(t), a function which is given by

$$x(t) = Ae^{\alpha t}$$

$$\frac{dx}{dt} = A\alpha e^{\alpha t} = \alpha x$$

$$\frac{d2x}{dt2} = A\alpha^2 e^{\alpha t} = \alpha^2 x$$

Putting these values in equation

$$\alpha^2x + 2\alpha^2\beta x + \omega_0^2x =0$$

$$\alpha^2 + 2\alpha^2\beta + \omega_0^2 =0 \text{ ………… (3)}$$

$\alpha = -\beta\pm\sqrt{\beta 2 -}\omega_0^2$, is the general solution of above quadratic equation.

As we know,

$$x(t) =A_1e^{\alpha t}+ A_2e^{\alpha t}$$

$$x(t) = A_1e^{(-\beta+\sqrt{\beta 2-}\ \omega 02)t}+ A_2e^{(-\beta-\sqrt{\beta 2-}\ \omega 02)t}$$

$$x(t) = e^{-\beta t}\ (A_1e^{\sqrt{\beta 2-}\ \omega 02\ t}+ A_2e^{-\sqrt{\beta 2-}\ \omega 02\ t}) \text{ … (4)}$$

Depending upon the strength of damping force the quantity $(\beta^2-\omega_0^2)$ can be positive /negative /zero giving rise to three different cases.

Case-1:- if $\beta<\omega_0^2$=> underdamping (oscillatory)

Case-2:- if $\beta>\omega_0^2$=> overdamping (non-oscillatory)

Case-3:- if $\beta=\omega_0^2 \Rightarrow$ critical damping (non-oscillatory)

Case-1: [Under damping $\omega_0^2>\beta^2$]

If $\beta^2<\omega_0^2$, then $\beta^2-\omega_0^2=$ -ve

let $\beta^2-\omega_0^2=-\omega^2_1 \Rightarrow \sqrt{\beta^2-\omega_0{}^2}=i\,\omega_1$

where $\omega_1=\sqrt{\omega_0^2-\beta^2}$ = Real quantity

So the general equation of damped oscillation/equation (IV) becomes

$X(t) = e^{-\beta t}(A_1e^{i\omega_1 t}+A_2e^{-i\omega_1 t})$

By setting

$A_1=r/_2e^{i\theta}$ and $A_2= r/_2e^{-i\theta}$,

$X(t)= e^{-\beta t}[r/_2e^{i(\theta+\omega_1 t)}+ r/_2\, e^{-i(\theta+\omega_1 t)}]$

$=re^{-\beta t}[e^{i(\theta+\omega_1 t)}+ e^{-i(\theta+\omega_1 t)}]/2$

$X(t)= re^{-\beta t}\cos(\theta+\omega_1 t)$………(v)

Here $\cos(\theta+\omega_1 t)$ represents the motion is oscillatory having angular frequency 'ω_1' .The constant 'r' and 'θ' are determined from initial potion & velocity of oscillatior

$T_1=2\pi/\omega_1$

$T_1=2\pi/\sqrt{\omega_0^2-\beta^2}$……(vi) (time period of damped oscillator)

$T_1>T$ (where T= time period of undamped oscillator

Implies $f_1< f$

Frequency of damped oscillator is less than that of the undamped oscillator.

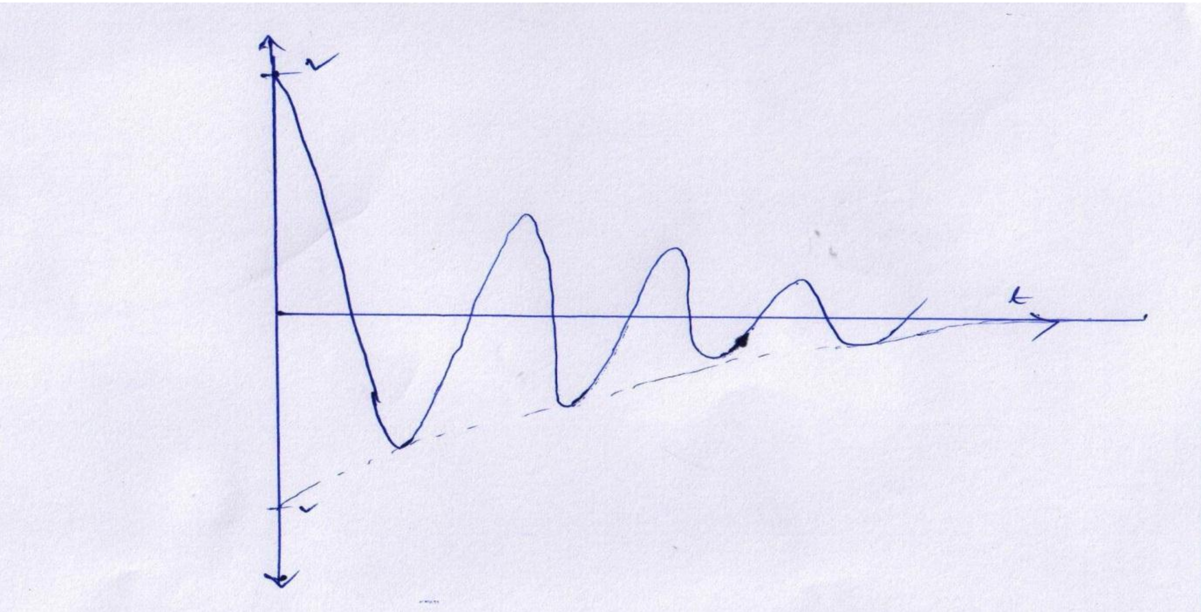
In under damped condition amplitude is no more constant and decreases exponentially with time, till the oscillation dies out.

Mean life time: The time interval in which the oscillation falls to 1/e of its initial value is called mean life time of the oscillator. (τ)

$1/_e\, a = a\, e^{-\beta\tau}m \Rightarrow e^{-\beta\tau_m} = \frac{1}{e}$,

$\Rightarrow -\beta\, \tau_m = \log_e 1/e$

$$\Rightarrow \tau_m = \frac{1}{\beta}$$



Velocity of underdamped oscillation:

$X(t) = re^{-\beta t}\cos(\omega_1 t + \theta)$

$\Rightarrow \frac{dx}{dt} = r[-\beta e^{-\beta t}\cos(\omega_1 t + \theta) - e^{-\beta t}\omega_1\, \mathbf{sin(}\ \omega_1 t + \theta)$

$\Rightarrow \frac{dx}{dt} = v = -re^{-\beta t}[\beta\cos(\omega_1 t + \theta) + \omega_1\, \mathbf{sin(}\ \omega_1 t + \theta) \ldots (vi)$

Now , x=0& t=0,

$X(t) = re^{-\beta t}\cos(\omega_1 t + \theta)$

$\Rightarrow 0 = re^{0}\cos(0 + \theta)$

$\Rightarrow 0 = \cos\theta$

$$\Rightarrow \theta = \frac{\pi}{2}$$

Using the value of θ & t=0 in the equation (vii) we have

v_0= -r ω_1

Where value of V_0 in ....

Calculation of Energy(instantaneous):

K.E = $\frac{1}{2}mv^2$

K.E = $\frac{1}{2}mv^2 e^{-2\beta t}[\beta^2\cos^2(\omega_1 t+ \theta)+ \omega_1^2\sin^2(\omega_1 t+ \theta)+ \beta\omega_1\sin^2(\omega_1 t+ \theta)]$

Potential Enegy:

P.E=$\frac{1}{2}kx^2$

$=\frac{1}{2}kr^2 e^{-2\beta t}\cos^2(\omega_1 t+ \theta)$

Total Energy:

T.E=K.E+P.E

$=e^{-2\beta t}[(\frac{1}{2}mv^2+\frac{1}{2}kr^2)\cos^2(\omega_1 t+\theta)+\frac{1}{2}mr^2\omega_1^2\sin^2(\omega_1 t+\theta)$
$+ \frac{1}{2}mv^2\ \beta\omega_1\sin2(\omega_1 t+ \theta)]$

Total average energy:

$< E > = \frac{1}{2}mr^2\ \omega_0^2 e^{-2\beta t}$

$=E_0 e^{-2\beta t}$

Where, E_0 =Total energy of free oscillation

The average energy decipated during one cycle

$< P(t) >$ =Rate of energy

$=\frac{d}{dt} < E >$

$=2\beta E$

Decrement

The decrement measures the rate at which amplitude dies away.

The ratio between amplitude of two successive maxima, is the decrement of the oscillator.

$re^{-\beta t} / re^{-\beta(t+T)} = re^{+\beta t}$

The logarithmic decrement of oscillator is 'λ'

$\lambda = \log_a e^{\beta t}$

$\Rightarrow \beta T = 2\pi\beta/\sqrt{\omega_0^{2} - \beta^2}$

$\Rightarrow \lambda = \log_a a_0/a_1 = \log a_1/a_2 = \dots\dots\dots\dots = e^{\beta t} = e^{\lambda 0}$

Rate of two amplitudes of oscillation whichare separated by one period

Relaxation time(τ):

It is the time taken by damped oscillation by decaying of its energy 1/e of its initial energy.

$\Rightarrow \frac{1}{e}\varepsilon_0 = \varepsilon_0 e^{-2\beta\tau}$

$\Rightarrow \frac{1}{e} = e^{-2\beta\tau}$

$\Rightarrow \text{Log}e^{-1} = \log e^{-2\beta\tau}$

$\Rightarrow -1 = -2\beta\tau$

$\Rightarrow \tau = 1/2\beta = m/b$

Case-II:(over damping oscillation)

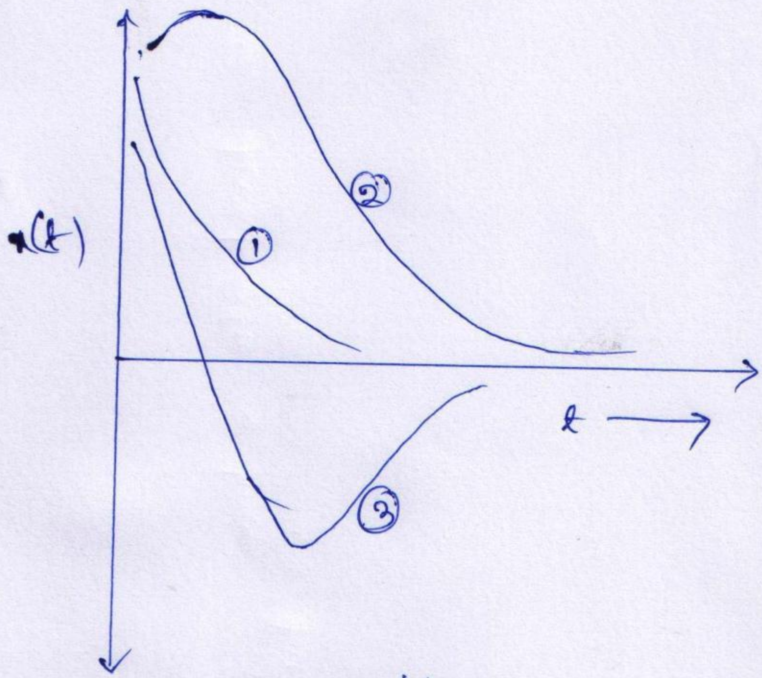
Here $\beta^2 > \omega_0^2$

$$\sqrt{\beta^2 - \omega_0^2} = +\text{ve quantity}$$

$= \alpha$ (say)

$$X(t) = e^{-\beta t}\,(A_1 e^{\alpha t} + A_2 e^{-\alpha t}) \ldots\ldots\ldots\ldots \text{(viii)}$$

Depending upon the relative values of α, β, A_1, A_2 & initial position and velocity the oscillator comes back to equilibrium position.



① Displacement decreases to '0'

② Displacement first increases, attains max. & then decreases to zero.

③ Displacement changes sign i.e the oscillator over that mean position, attain max. & comes back mean position.

The motion of simple pendulum in a highly viscous medium is an example of over damped oscillation.

Quality factor:

$$Q=2\pi.\frac{\text{Energy stored in system}}{\text{Energy loss per period}} = 2\pi.\frac{<E>}{<P>T} = \frac{2\pi}{T}.\tau$$

$$\Rightarrow Q = \omega\tau$$

Critical damping:

$\beta^2 = \omega_0^{\,2}$

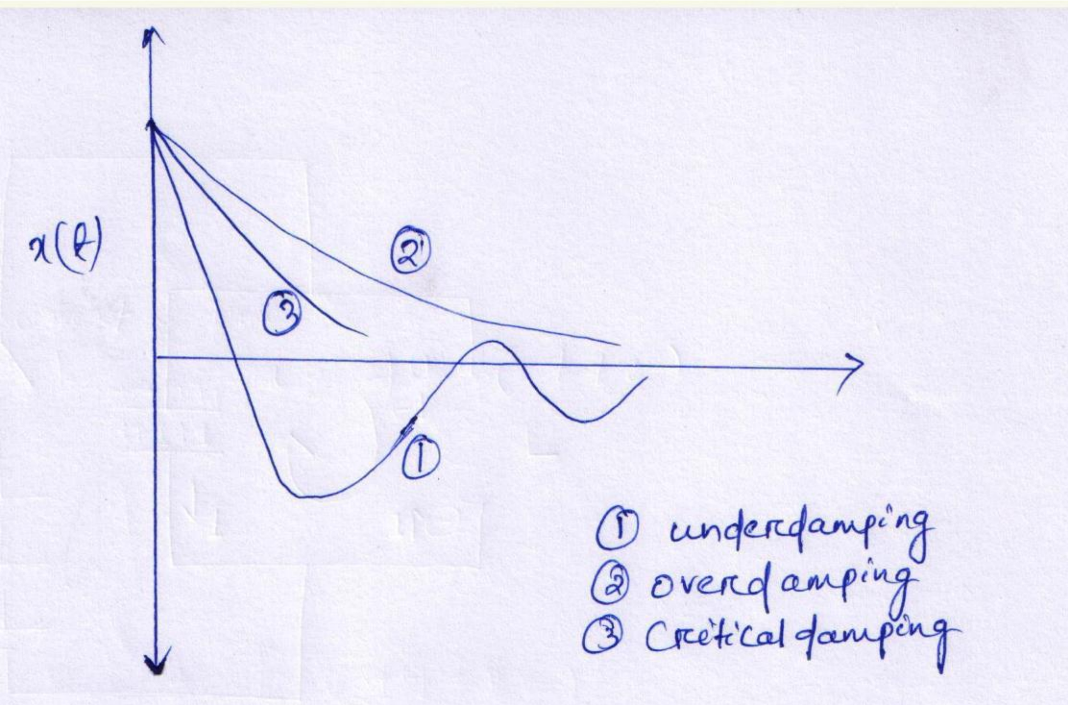
The general solution of equation (ii) in this case,

$X(t) = (Ct+D)\, e^{-\beta t}$ …………………………………(ix)

Here the displacement approaches to zero asymptotically for given value of initial position and velocity a critically damped oscillator approaches equilibrium position faster than other two cases.

Example: The springs of automobiles or the springs of dead beat galvanometer.

Curves of three Cases:



Forced Oscillation

The oscillation of a oscillator is said to be forced oscillator or driven oscillation if the oscillator is subjected to external periodic force.

If an external periodic sinusoidal force 'Fcosωt' acts on a damped oscillator, its equation of motion is written as,

$$F_{net} = -kx - b\,\frac{dx}{dt} + F\cos\omega t$$

$$m\,\frac{d^2x}{dt^2} + = -kx - b\,\frac{dx}{dt} + F\cos\omega t$$

$$\frac{d^2x}{dt^2} + +\frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F}{m}\cos\omega t$$

$$\boxed{\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = f_0\cos\omega t}$$

-- (i)

Where $\beta = \frac{b}{2m}$, $\omega_0^2 = \frac{k}{m}$ and $f_0 = \frac{F}{m}$, and β and ω_0^2 respectively called as damping coefficient, natural frequency.

Equation (i) is also represented as

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t}$$

Equation (i) represents the general equation of forced oscillation.

Equation (i) is a non-homogenous differential equation with constant co-efficient. For weak damping ($\omega_0^2 > \beta^2$), the general equation contains,

$x(t) = x_c(t) + x_p(t)$

Where $x_c(t)$ is called complementary solution and its value is

$x_c(t) = e^{-\beta t}\,(A_1 e^{\sqrt{\beta^2-\omega_0^2}} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}})$ ………………(ii)

Now $x_p(t)$ is called the particular integral part.

Let us choose

$x_p(t) = P\cos(\omega t - \delta)$

$\dot{x}(t) = -P\omega\sin(\omega t - \delta)$

$\ddot{x}(t) = -P\omega^2\cos(\omega t - \delta)$……………………….(iii)

Putting $x_p(t)$, $\dot{x}(t)$, $\ddot{x}(t)$ in eq^n (i) we get

$-P\omega^2\cos(\omega t-\delta) - 2\beta P\omega\sin(\omega t-\delta) + \omega_0^2 P\cos(\omega t-\delta) = f_0\cos\omega t$

$-P\omega^2\cos(\omega t-\delta) - 2\beta P\omega\sin(\omega t-\delta) + \omega_0^2 P\cos(\omega t-\delta) = f_0\cos(\omega t-\delta+\delta)$

$- P\omega^2\cos(\omega t-\delta)-2\beta P\omega \sin(\omega t-\delta) + \omega_0^2 P\cos(\omega t-\delta) = f_0 [\cos (\omega t-\delta) .\cos \delta - \sin(\omega t-\delta) .\sin\delta]$

Now, compairing the coefficient of cos(ωt-δ) and sin (ωt-δ) on both sides,

$(\omega_0^2-\omega^2)P = f_0\cos\delta$ ……………………………(iv)

$2\beta P\omega = f_0\sin\delta$ ……………………………………(v)

Squaring and adding eq^n (iv) & (v)

$\{(\omega_0^2-\omega^2)P\}^2 +4 \beta^2 P^2\omega^2 =f_0^2$

$$P = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}} \quad \text{…………………(vi)}$$

Now dividing eq^n (v) by (iv)

$$\delta= \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right) \quad \text{…………………..(vii)}$$

$$x_p = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}\cos(\omega t-\delta) \quad \text{(steady state solution)}$$

Now, $x(t) = x_c(t) + x_p(t)$

$$x(t) = e^{-\beta t}\left(A_1e^{\sqrt{\beta^2-\omega_0{}^2}}+ A_2e^{-\sqrt{\beta^2-\omega_0{}^2}}\right) + \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}\cos(\omega t-\delta)$$

Steady state behavior:

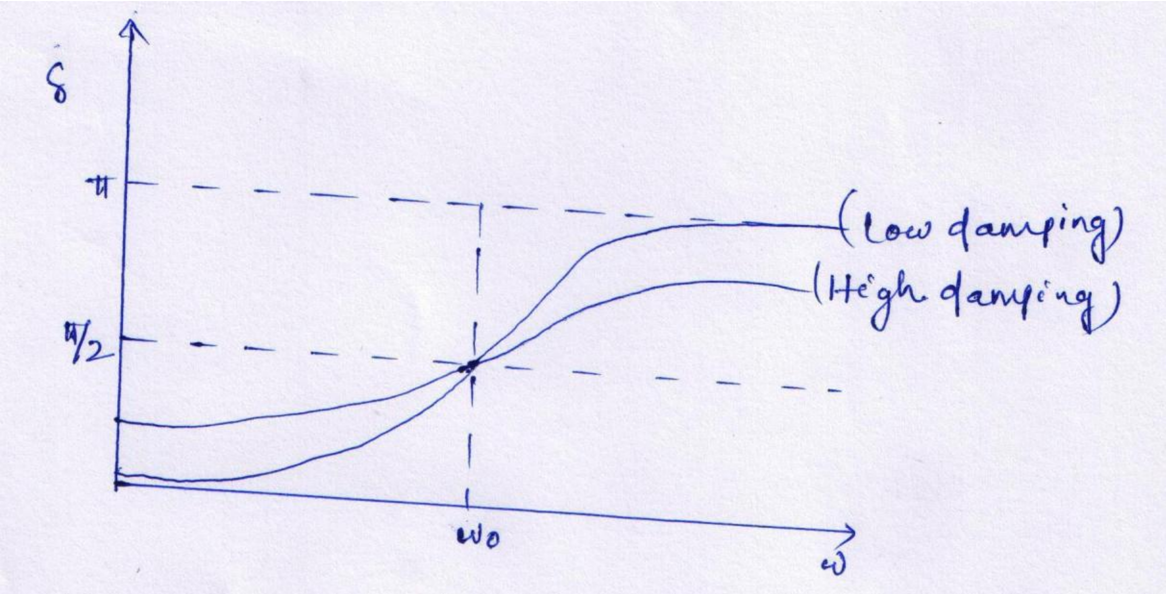
Frequency:-The Oscillator oscillates with the same frequency as that of the periodic force.

$\omega_0$and ω are very close to each other then beats will be produced and these beats are transient as it lasts as long as the steady state lasts. The duration between transient beats is determined by the damping coefficient 'β'.

Phase: The phase difference 'δ' between the oscillator and the driving force or between the displacement and driving is

$$\delta=\tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2-\omega^2}\right)$$

This shows that there is a delay between the action of the driving force and response of the oscillator.



(In the above figure $f_Q=\omega_0$ and $f_p=\omega$)

At $\omega=\omega_0$, $\varphi=\frac{\pi}{2}$, the displacement of the oscillator lags behind the driving force by $\frac{\pi}{2}$.

At $\omega<<\omega_0$ then $\delta=0\rightarrow\delta=0$

For $\omega>>\omega_0$ then $\delta=-\frac{2p}{\omega}\rightarrow -0=\pi$

Amplitude: The amplitude of driven oscillator , in the steady state , is given by

$$A=\frac{F/m}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\beta^2\omega^2}}=\frac{f_0}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\beta^2\omega^2}}$$

It depends upon (ω_0^2-ω^2). If it is very small, then the amplitude of forced oscillation increases.

<u>Case-1:</u> At very high driving force i.e $\omega >> \omega_0$ and damping is small (β is small) or ($\beta \to 0$)

$$A = \frac{f_0}{\sqrt{\omega^4 + 0}}$$

$$A = \frac{f_0}{\omega^2}$$

$$\boxed{A = \frac{F}{m\omega^2}}$$

Amplitude is inversely proportional to the mass of the oscillator & hence the motion is mass controlled motion.

<u>Case-2:</u> At very low driving force ($\omega << \omega_0$) and damping is small

($\beta \to 0$),

i.e. $\omega_0^2 - \omega^2 \cong \omega_0^2$

$$A = \frac{f_0}{\sqrt{\omega_0^4}}$$

$$A = \frac{f_0}{\omega_0^2}$$

$$\boxed{A = \frac{F}{m\omega_0^2}}$$

So, when the low driving force is applied to oscillator, the amplitude remains almost constant for low damping. The amplitude of the forced oscillator in the region $\omega << \omega_0$ and $\beta < \omega_0$ is inversely proportional to the stiffness constant (k) and hence motion is called the stiffness controlled motion.

Case:-iii (Resistance controlled motion)

When angular frequency of driving force=natural frequency of oscillator i.e.($\omega=\omega_0$)

$A=f_0/\sqrt{4\beta^2\omega^2}=f_0/2\beta\omega$

$A=f/b\omega=f/b\omega_0$

RESONANCE:-

The amplitude of vibration becomes large for small damping(β is less) and the maximum amplitude is inversely proportional to resistive term (b) hence called as resonance. It is the phenomenon of a body setting a body into vibrations with its natural frequency by the application of a periodic force of same frequency.

If the amplitude of oscillation is maximum when the driving frequency is same as natural frequency of oscillator. (I.e. $\omega=\omega_0$).

'A' will be the max. Only the denominator of the expression

$\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$ is minimum i.e.

$\frac{d}{d\omega}[\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}=0$

$\Rightarrow -4\omega\omega_0^2+4\omega^3+8\beta^2\omega=0$

$\Rightarrow -\omega_0^2+\omega^2+2\beta^2=0$

$\Rightarrow \omega=\sqrt{(\omega_0^2-2\beta^2)}=\omega_0\sqrt{(1-2\beta^2/\omega_0^2)}$

It is the value of angular frequency, where 'A' will be maximum in presence of damping force

But when damping is very small,

$\omega=\omega_0\ (\beta\rightarrow 0)$

The max value of 'A 'when damping is present

$A = f_0/\sqrt{[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]}$

$= f_0/\sqrt{[\omega_0^2 - (\omega_0^2 - 2\beta^2)]2 + 4\beta(\omega_0^2 - 2\beta^2)}$

$= f_0/\sqrt{4\beta^4 + 4\beta^2\omega_0^2 - 8\beta^4}$

$= f_0/\sqrt{[4\beta^2\omega_0^2 - 4\beta^4]}$

$A_{max} = f_0/2\beta\sqrt{(\omega_0^2 - \beta^2)} = f/2m\beta\sqrt{(\omega_0^2 - \beta^2)}$

This is called amplitude Resonance.

Value of the frequency at which amplitude resonance occurs i.e. amplitude becomes maximum.

$B_1 < \beta_2 \quad f_r = \omega/2\pi$

$= \sqrt{(\omega^2_0 - 2\beta^2)}/2\pi$

Damping is small,

$f_r = \omega_0/2\pi$

Here, fr' is called resonant frequency.

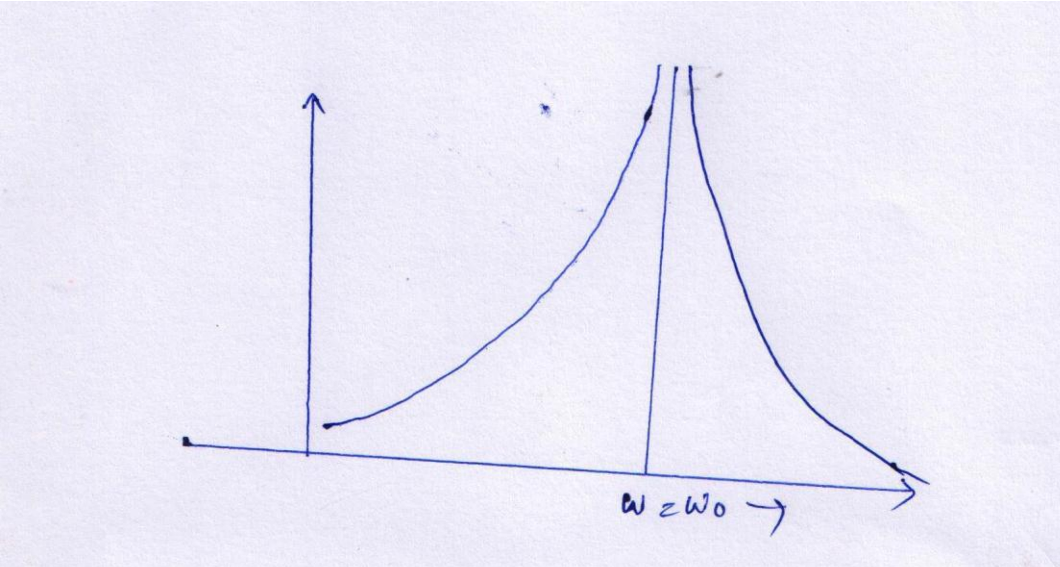
Phase at resonance:-

$\Phi = \pi/2$

Velocity of oscillator is in same phase with the driving force .Therefore, the driving force always acts in the direction of motion of oscillator. So energy transfers from driving force to oscillation are max^{im}.

Sharpness of resonance:-

The amplitude is maximum at resonance frequency which decreases rapidly as the frequency increases or decreases from the resonant frequency.



The rate at which the amplitude decreases with the driving frequency on either side of resonant frequency is termed as ''sharpness of resonance''.

Different condition:-

(i) For $\omega=\omega_0\pm\beta$, the amp. Becomes $A=A_{max}/\sqrt{2}$.The width of resonance curve i.e. the range of frequency over which the amplitude remains more than $A_{max}/\sqrt{2}$.

$\Delta\omega= (\omega_0+\beta)-(\omega_0-\beta)=2\beta$

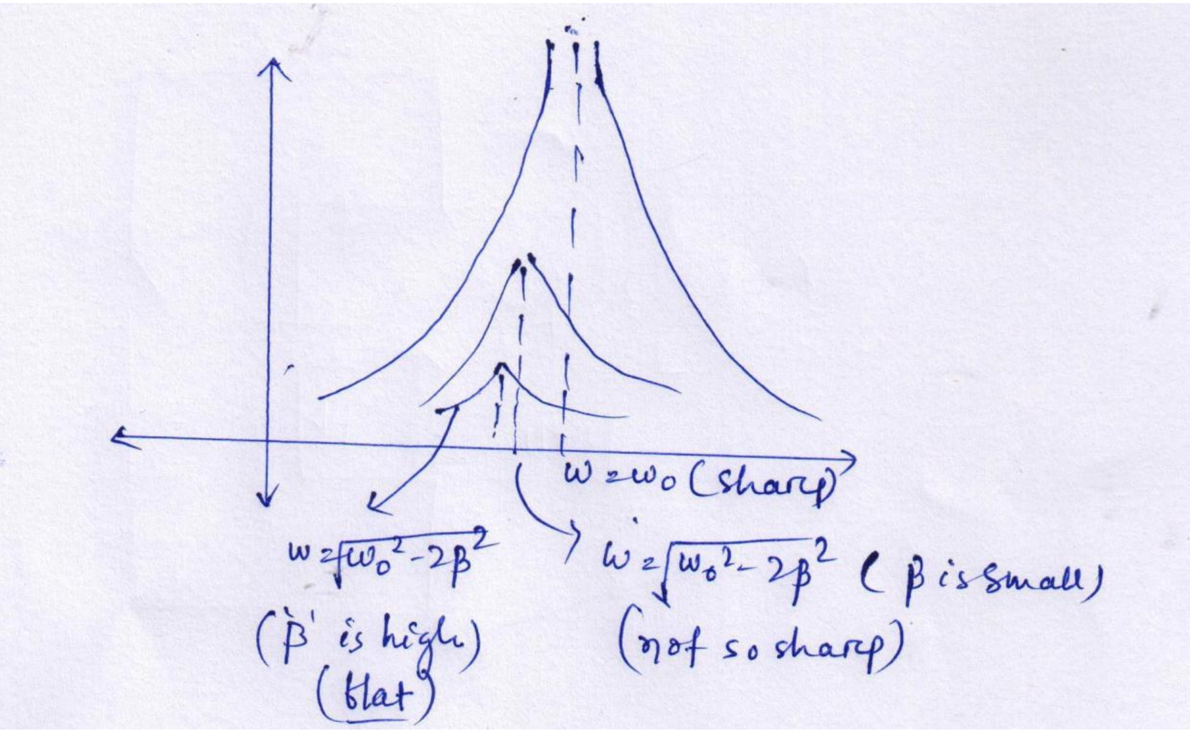
Thus if β' is small,$\Delta\omega$ is small.

(ii) For $\beta=0$, $A\rightarrow\infty$ at $\omega=\omega_0$.

(iii)If there is small 'β', amp. Resonance occurs lesser value amp is max at $\omega=\omega_r$.

(iv)If β''is high, A' is max but the peak moves towards left &max. amp decreases.

(v)So resonance is sharp for low 'β' & flat for high 'β'.



Velocity:-

$$X=x_p=f_0/\sqrt{[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2]}\cos(\omega t-\delta)$$

$$V=-\omega f_0/\sqrt{[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2]}\sin(\omega t-\delta)$$

$$\boxed{V=\omega f_0/\sqrt{[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2]}\sin\left(\omega t-\delta+\frac{\pi}{2}\right)}$$

$$V_{max}=\omega f_0/\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$$

*here also 'v' is max. When $\omega_0=\omega$.

$(V_{max.amp})=f/b$

Calculation of energy:-

$$x=\frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}\cos(\omega t-\delta)$$

Where $\frac{f_0}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\beta^2\omega^2}}$=A

V=$\frac{f_0\omega}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\beta^2\omega^2}}$cos(pt-δ+π/2)

Average potential energy:-

P.E=$\frac{1}{2}$ kx^2

=$\frac{1}{2}$ k$\frac{f_0^2}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\beta^2\omega^2}}$cos^2(ωt-δ)

=$\frac{1}{2}$ kA^{2}cos^2(ωt-δ)

<P.E>=$\frac{1}{4}$ kA^2

=$\frac{1}{4}$mω$_0{}^2$A^2

(average of cos^{2}θ=1/2)

Average kinetic energy:-

K.E=$\frac{1}{2}$ mv^2

=$\frac{1}{2}$m $\frac{f_0^2\omega^2}{\left(\omega_0^2-\omega^2\right)^2+4\beta^2\omega^2}$cos^2(ωt-δ+π/2)

=$\frac{1}{2}$ mω^{2}A^{2}cos^2(pt-δ+π/2)

<K.E>=$\frac{1}{4}$ mω^{2}A^2

Total average energy:-

$<\in> = < K.E > + < P.E >$

$= \frac{1}{4}\,\mathrm{m}\omega^2\mathrm{A}^2 + \frac{1}{4}\,\mathrm{m}\omega_0{}^2\mathrm{A}^2$

$$\boxed{\langle \epsilon \rangle = \frac{1}{4} mA^2 (\omega_0{}^2 + \omega^2)}$$

POWER:-

i). Power absorption:

$$P_{ab} = F_{P_e} . v$$

$$= F \operatorname{Cos} (\omega t - \delta) . \frac{f_0 \, \omega}{\sqrt{(\omega_0{}^2 - \omega^2)^2 + 4\beta^2 \omega^2}} \operatorname{Cos} (\omega t - \delta + {}^{\pi}/_{2})$$

$$= A \, F \operatorname{Cos} (\omega t - \delta) \operatorname{Sin} (\omega t - \delta)$$

$$= \frac{1}{2} AF \operatorname{Sin} 2 (\omega t - \delta)$$

$$< P_{abs} > = \frac{f_0{}^2 \, \beta \omega^2}{[(\omega_0{}^2 - \omega^2)^2 + 4\beta^2 \omega^2]}$$

$$= m \, A^2 \beta \, \omega^2$$

* P_{max}absorbed when $\omega_o = \omega$

$$\Rightarrow P_{max} = \frac{f_0{}^2 \cancel{\beta \omega_0{}^2}}{4\cancel{\beta^2 \omega_0{}^2}} = \frac{f_0{}^2}{4\beta}$$

ii). Power dissipation:

$$P_{dis} = F_{damp} . v \quad \text{Or} \quad F_{resistive} \times \text{Inst. velocity}$$

$$= + b . \frac{dx}{dt} . v$$

$$= + b \, v^2$$

$$P_{dis} = 2m \, \beta \, v^2 \qquad (\beta = {}^{b}/_{2m})$$

$$\Rightarrow P_{dis} = 2m\beta . \frac{f_0{}^2 \omega^2}{[(\omega_0{}^2 - \omega^2)^2 + 4\beta^2 \omega^2]} . \operatorname{Cos}^2 \left(pt - \delta + \frac{\pi}{2}\right)$$

$\Rightarrow < P_{dis} > = \cancel{2} m\beta A^2 \omega^2 \times {}^{1}/_{\cancel{2}}$

$\Rightarrow < P_{dis} > = m\beta A^2 \omega^2$

Thus in the steady state of forced vibration, the average rate of power supplied by the forcing system is equal to the average of work done by the forced system against the damping force.

<u>QUALITY FACTOR:-</u>

Quality factor is a measure of sharpness of resonance.

Q- Factor is defined as,

$$Q = 2\pi \times \frac{\text{average energy stored per cycle}}{\text{average energy dissipated per cycle}}$$

$$= 2\pi \times \frac{E_{av}}{T.P_{av}}$$

$$= 2\pi \times \frac{\frac{1}{4}\cancel{m}\,\cancel{A^2}(\omega_0{}^2 + \omega^2)}{T \times \cancel{m}\, \beta\omega^2 \cancel{A^2}}$$

$$= \frac{\omega(\omega_0{}^2 + \omega^2)}{4\beta\omega^2} = \frac{(\omega_0{}^2 + \omega^2)}{4(\beta\omega)}$$

At $\omega = \omega_0$, for weak damping

$$Q = \frac{2\omega_0{}^2}{4(\beta\omega_0)}$$

$\Rightarrow$ $\boxed{Q = \frac{\omega_0}{2\beta}}$

βSmall, Q → Large, sharpness of resonance is more.

Again,

$$\mathbf{Q} = \frac{\textbf{Resonant frequency}}{\textbf{width of resonance curve}} = \frac{\omega_0}{2\beta}$$

Larger value of Quality factor (lessβ), sharper is the resonance.

System	Q value
Earthquake	250 – 1400
Violin string	10^3
Microwave resonator	10^5
Crystalosill	10^6
Excetetation	10^8

Amplitude Resonance	Velocity Resonance

1. In amp. Resonance, the amp. of oscillator is maximumfor a particular frequency of the applied force. 2. Amplitude resonance occurs at $\omega_r = (\omega_0 - 2\beta^2)^{1/2}$ 3. At applied frequency $\omega = 0$, the amp. of the freq. oscillator is F/k 4. The phase of the forced oscillator with respect to that of applied force is $\pi/2$	1. The velocity amplitude of the forced oscillator is the maximum at a particular frequency of applied force. 2. Velocity resonance occurs at $\omega = \omega$ 3. Applied frequency $\omega = 0$, the velocity amplitude is zero. 4. Phase of the forced oscillator with respect to that of applied force is ….

Mechanical Impedance

The force required to produce unit velocity is called the mechanical impedance of the oscillator.

$$z = F/v$$

$$x = A\, e^{i(\omega t - \varphi)}$$

$$v = \frac{dx}{dt} = \omega i A e^{i(\omega t - \varphi)}$$

$$\Rightarrow z = \frac{F\, e^{i\omega t}}{i\omega A e^{i(\omega t - \varphi)}} = \frac{F}{i\omega A e^{-i\varphi}} = \frac{mf}{i\omega A e^{-i\varphi}}$$

$$= \frac{mf}{i\omega \frac{f}{\left((\omega_0{}^2 - \omega^2)^2 + 2\beta\omega i\right)}}$$

$$= 2\beta m - \frac{im(\omega_0{}^2 - \omega^2)}{\omega}$$

$$z^* = 2\beta m + \frac{im(\omega_0{}^2 - \omega^2)}{\omega}$$

$$|z| = z^* z = m\left[4\beta^2 + \frac{1}{\omega^2}(\omega_0{}^2 - \omega^2)^2\right]^{1/2}$$

$$A = \frac{F}{\omega|z|}$$

For a particular ω, $\boxed{A \propto \frac{1}{|z|}}$

INTERFERENCE

Coherent Superposition:

The superposition is said to be coherent if two waves having constant phase or zero phase difference.
In this case, the resultant intensity differs from the sum of intensities of individual waves due to interfering factor.
i.e. $I \neq I_1 + I_2$

Incoherent Superposition:

The superposition is said to be incoherent if phase changes frequently or randomly.
In this case, the resultant intensity is equal to the sum of the intensities of the individual waves.
i.e. $I = I_1 + I_2$

Two Beam Superposition:

When two beam having same frequency, wavelength and different in amplitude and phase propagates in a medium, they undergo principle of superposition which is known as two beam superposition.

Let us consider two waves having different amplitude and phase are propagated in a medium is given as

$$\psi_1 = a_1 \sin(kx - \omega t + \varphi_1) \quad (1)$$

$$\psi_2 = a_2 \sin(kx - \omega t + \varphi_2) \quad (2)$$

Applying the principle of superposition

$$\psi = \psi_1 + \psi_2$$

$$\psi = a_1 \sin(kx - \omega t + \varphi_1) + a_2 \sin(kx - \omega t + \varphi_2)$$

$$= a_1 \sin(kx - \omega t) \cos \varphi_1 + a_2 \cos(kx - \omega t) \sin \varphi_1 + a_2 \sin(kx - \omega t) \cos \varphi_2 + a_2 \cos(kx - \omega t) \sin \varphi_2$$

$$= (a_1 \cos \varphi_1 + a_2 \cos \varphi_2) \sin(kx - \omega t) + (a_1 \sin \varphi_1 + a_2 \sin \varphi_2) \cos(kx - \omega t) \quad (3)$$

Let

$$a_1 \cos \varphi_1 + a_2 \cos \varphi_2 = A \cos \theta \quad (4)$$

and $a_1 \sin \varphi_1 + a_2 \sin \varphi_2 = A \sin \theta$ (5)

$$\psi = A \cos \theta \sin(kx - \omega t) + A \sin \theta \cos(kx - \omega t)$$

$$= A[\sin(kx - \omega t) \cos \theta + \cos(kx - \omega t) \sin \theta]$$

$$\psi = A \sin(kx - \omega t + \theta) \quad (6)$$

Squaring and adding equation (4) and (5)

$$A^2 = (a_1 \cos \varphi_1 + a_2 \cos \varphi_2)^2 + (a_1 \sin \varphi_1 + a_2 \sin \varphi_2)^2$$

$$= a_1^2 cos^2 \varphi_1 + a_2^2 cos^2 \varphi_2 + 2a_1 a_2 \cos \varphi_1 \cos \varphi_2 + a_1^2 sin^2 \varphi_1 + a_2^2 sin^2 \varphi_2 + 2a_1 a_2 \sin \varphi_1 \sin \varphi_2$$

$$A^2 = a_1^2 + a_2^2 + 2a_1 a_2 [\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2]$$

$$A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\varphi_2 - \varphi_1)$$

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\varphi_2 - \varphi_1)} \quad (7)$$

We know, $I \propto A^2$

$$\Rightarrow I = KA^2$$

$$= K(a_1^2 + a_2^2 + 2a_1 a_2 \cos(\varphi_2 - \varphi_1)$$

$$\Rightarrow I = Ka_1^2 + Ka_2^2 + K2a_1 a_2 \cos(\varphi_2 - \varphi_1)$$

$$\Rightarrow I = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2} \cos(\varphi_2 - \varphi_1) \quad (8)$$

Dividing equation (5) by (4), we get,

$$\tan\theta = \frac{a_1 \sin\varphi_1 + a_2 \sin\varphi_2}{a_1 \cos\varphi_1 + a_2 \cos\varphi_2}$$

Coherent Superposition:

In coherent superposition, the phase difference remains constant between two beams.

i.e. $\cos(\varphi_2 - \varphi_1) = 1 or - 1$

$If\ \cos(\varphi_2 - \varphi_1) = 1$

Now equation (7) and (8) becomes,

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2}$$

$$\Rightarrow A = (a_1 + a_2)^2$$

$\Rightarrow A_{max} = a_1 + a_2$ and $\quad I = I_1 + I_2 + 2\sqrt{I_1 I_2}$

$$\Rightarrow I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2$$

The two beams having same amplitude,

$$a_1 = a_2 = a$$

$$\Rightarrow A_{max} = 2a$$

$$\Rightarrow I_{max} = 4I_0$$

Again, if $\cos(\varphi_2 - \varphi_1) = -1$

$$A_{min} = \sqrt{a_1^2 + a_2^2 - 2a_1 a_2}$$

$$\Rightarrow A_{min} = a_1 - a_2$$

$$I = I_1 + I_2 - 2\sqrt{I_1 I_2}$$

$$I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2$$

For same amplitude,

$$A_{min} = 0$$

$$\Rightarrow I_{min} = 0$$

Incoherent Superposition:

In incoherent superposition the phase difference between the waves changes frequently or randomly, so the time average of the interfering term$(2\sqrt{I_1 I_2}\ \cos(\varphi_2 - \varphi_1))$ vanishes as the cos value varies from -1 to 1.

Here, $A = \sqrt{a_1^2 + a_2^2}$

$\Rightarrow I = I_1 + I_2$

Multiple beam superpositions:

When a number of beams having same frequency, wavelength and different amplitude and phase are undergoing the superposition, such superposition is known as multiple beam superpositions.

Let ψ_1,ψ_2,ψ_3, ψ_4.............ψ_n be the number of beams having same frequency, wavelength and different in amplitude and phase are propagating in a medium are given as,

$\psi_1 = A_1 \sin(kx - \omega t + \varphi_1)$

$\psi_2 = A_2 \sin(kx - \omega t + \varphi_2)$

:

:

:

$\psi_N = A_N \sin(kx - \omega t + \varphi_N)$

According to principle of superposition,

$\psi = \psi_1$,ψ_2,ψ_3, ψ_4.............ψ_N

$$= \sum_{i=1}^{N} \psi_i$$

$$\Rightarrow \psi = \sum_{i=1}^{N} A_i \sin(kx - \omega t + \varphi_i)$$

$\Rightarrow \psi = \sum_{i=1}^{N} A_i \sin(kx - \omega t + \varphi_i)$ (1)

where A_i = resultant amplitude of the i^{th} component.

φ_i =Phase of the i^{th} component.

$A \sin \varphi = \sum_{i=1}^{N} A_i \sin \varphi_i$ (2)

$A \cos \varphi = \sum_{i=1}^{N} A_i \cos \varphi_i$ (3)

Squaring and adding (2) and (3) we get,

$$A^2 = \sum_{i=1}^{N} A_i^2 + 2 \sum_{\substack{i=1 \\ i \neq j}}^{N} A_i A_j \cos(\varphi_j - \varphi_i)$$

The phase angle is given as,

$$\tan \varphi = \frac{\sum_{i=1}^{N} A_i \sin \varphi_i}{\sum_{i=1}^{N} A_i \cos \varphi_i}$$

Coherent Superposition:

In this case the phase difference between the waves remains constant i.e. $(\cos \varphi_j - \varphi_i) = +1$

$$\Rightarrow A^2 = \sum_{i=1}^{N} A_i^2 + 2 \sum_{\substack{i=1 \\ i \neq j}}^{N} A_i A_j$$

If all the beams having equal amplitudes.

i.e.$A_1 = A_2 = \cdots = A_N = A_1$

$$\Rightarrow A^2 = (NA_1)^2 = N^2 A_1^2$$

Now, $I = kA^2$

$$\Rightarrow I = kN^2 A_1^2$$

$$\Rightarrow I_{coherent} = N^2 I_1$$

Incoherent Superposition

In incoherent superposition, the phase difference between the beams changes frequently or randomly due to which the time average of factor $< \sum_{i=1}^{N} A_i A_j \cos(\varphi_j - \varphi_i) >$ vanishes as cos value varies from-1 to +1

$$\sum_{i=1}^{N} A_i A_j \cos(\varphi_j - \varphi_i) = 0$$

$$A^2 = N \sum_{i=1}^{N} A_i^2$$

Now , $I_{incoherent} = KA^2$

$= KN \sum_{i=1}^{N} A_i^2$

$= KNA_1^2$

$\Rightarrow I_{incoherent} = NI_1 \qquad \Rightarrow N = \frac{I_{coherent}}{I_{icoherent}}$

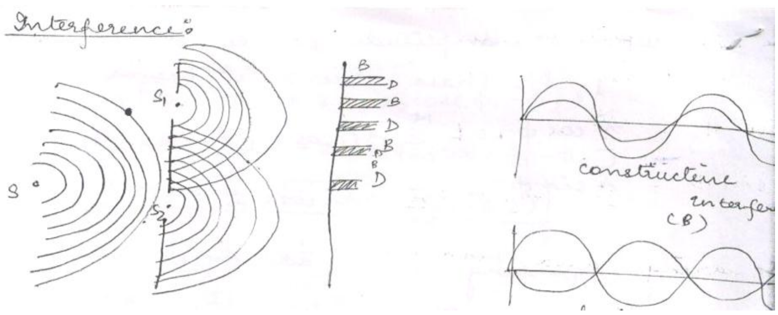
Interference:

The phenomenon of modification in distribution of energy due to superposition of two or more number of waves is known as interference.

To explain the interference, let us consider a monochromatic source of light having wavelength λ and emitting light in all possible directions.

According to Huygens's principle, as each point of a given wavefront will act as centre of disturbance they will emit secondary wave front on reaching slit S_1 and S_2.

As a result of which, the secondary wave front emitted from slit S1 and S2 undergo the Principle of superposition.



During the propagation, the crest or trough of one wave falls upon the crest and trough of other wave forming constructive interference, while the crest of one wave of trough of other wave producing destructive interference.

Thus, the interfering slit consisting of alternate dark and bright fringes, which explain the phenomenon of interference.

Mathematical treatment:

Let us consider two harmonic waves of same frequency and wavelength and different amplitude and phase are propagating in a medium given as

$$Y = y_1 + y_2$$

$$= \mathrm{a}\sin \omega t + b \sin(\omega t + \varphi)$$

$$= a \sin \omega t + b \sin \omega t \cos \varphi + b \cos \omega t \sin \varphi$$

$$= (a + b \cos \varphi) \sin \omega t + b \sin \varphi \cos \omega t$$

Let $a + b \cos \varphi = A \cos \theta$

$b \sin \varphi = A \sin \theta$

$y = A \cos \theta \sin \omega t + A \sin \theta \cos \omega t$

$y = A \sin(\omega t + \theta)$

Squaring and adding (2) and (3)

$A^2 cos^2\theta + A^2 sin^2\theta = (a + b \cos \varphi)^2 + b^2 sin^2\varphi$

$A^2 = a^2 + b^2 + 2ab \cos \varphi$

$\left[A = \sqrt{a^2 + b^2 + 2ab \cos \varphi}\right]$

As, $I \propto A^2$

$I = KA^2$

$= K(a^2 + b^2 + 2ab\cos\varphi)$

$\left[I = I_1 + I_2 + \sqrt{I_1} + \sqrt{I_2}\cos\varphi\right]$

Dividing equation (3) by (2) we get,

$$\tan\theta = \frac{b\sin\varphi}{a + b\cos\varphi}$$

Condition for maxima:

The intensity will be maximum when the constructive interference takes place i.e.

$\cos\varphi = +1$

$\cos\varphi = \cos 2n\pi$

$\varphi = \pm 2n\pi$, n=0, 1, 2...

$$\Rightarrow \frac{2\pi}{\lambda} \times path\ difference(\Delta x) = \pm 2n\pi$$

$[\Delta x = \pm n\lambda]$

$$\left[\Delta x = 2n\frac{\lambda}{2}\right]$$

The constructive interference is when φ difference is even multiple of π or integral multiple of 2π and path difference is an integral multiple of $\frac{\lambda}{2}$.

Now, $\left[I = I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}\right]$

$\left[I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2\right]$

$[A_{max} = a + b]$

If the waves having equal amplitude,

$[A_{max} = 2a]$

$I_{max} = KA_{max}^2$

$= K(2a)^2$

$= K4a^2$

$[I_{max} = 4I_0]$

Condition for minima

The intensity will be minimum destructive interference takes place

i.e. $\cos\varphi = -1$

$[\varphi = \pm(2n + 1)\pi]$ Where n = 0, 1, 2, 3...

$$\Rightarrow \frac{2\pi}{\lambda} \times (\Delta x) = \pm(2n + 1)\pi$$

$$\left[\Delta x = \pm(2n + 1)\frac{\lambda}{2}\right]$$

Thus destructive interference takes place when phase difference is odd multiple of π and path difference is odd multiple of $\frac{\lambda}{2}$.

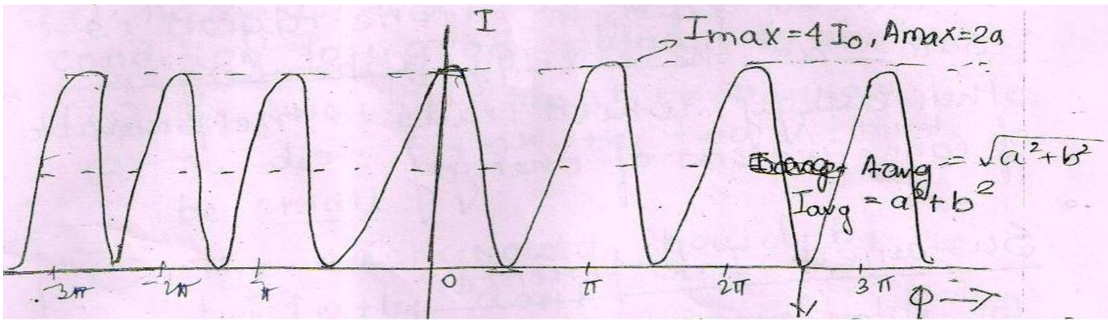
Now, $\left[I = I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2}\right]$

$\left[I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2\right]$

$[A_{min} = a - b]$

Intensity distribution curve

If we plot a graph between phase difference or path difference along X-axis and intensity along Y-axis, the nature of the graph will be symmetrical on either side.



$$I_{ave} = \frac{I_{max} + I_{min}}{2} = 2I_0$$

From the graph, it is observed that,

1) The fringes are of equal width
2) Maxima having equal intensities
3) All the minima's are perfectly dark

The phenomenon of interference tends to conservation of energy i.e. the region where intensity is 0, actually the energy present is maxima. As the minima's and maxima position changes alternatively so the disappearance of energy appearing is same as the energy appearing in other energy which leads to the principle of conservation of energy.

Sustained Interference

The interference phenomenon in which position of the maxima and minima don't changes with time is called sustained interference.

Condition for Interference

1) The two waves must have same frequency and wavelength.
2) The two source of light should be coherent.
3) The amplitude of wave may be equal or nearly equal.

Condition for good Contrast

I. The two slit must be narrow.
II. The distance between the two slit must be small.
III. The background should be perfectly dark.
IV. The distribution between the slit and the screen should be large.
V. The two waves may have equal or nearly equal amplitude (for sharp superposition).

Coherent Sources

The two sources are said to be coherent if they have same phase difference, zero phase difference or their relative phase is constant with respect to time.

Practical resolution of Coherent

Coherent sources from a single source of light can be realised as follows

A narrow beam of light can be split into its number of component waves and multiple reflections.

Component light waves are allowed to travel different optical path so that they will suffer a path difference and hence phase difference.

$$\left[phase\ difference = \frac{2\pi}{\lambda} \times path\ difference\right]$$

Methods for producing coherent sources/Types of interferences

Coherent sources can be produced by two methods

1) Division of wave front
2) Division of amplitude

Division of Wave front

The process of coherent source or interference by dividing the wave front of a given source of light is known as division of wave front.

This can be done by method of reflection or refraction. In this case a point source is used.

Examples

1. YDSE
2. Lylord's single mirror method
3. Fresnel's bi-prism
4. Bilet splitting lens method

DIVISION OF AMPLITUDE

The process of obtaining a coherent source by splitting the amplitude of light waves is called division of amplitude which can be done by multiple reflections.

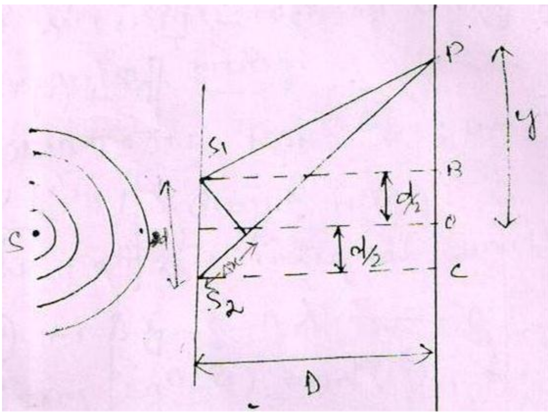
In this case, extended source of light is used.

1.Newton’s ring method

2.Thin film method

3. Michelson’s interferometer

Young’sDouble Slit Experiment:



In 1801 Thomas Young demonstrated the phenomenon of interference in the laboratory with a suitable arrangement. It is based on the principle of division of wavefront of interference. The experiential arrangement consists of two narrow slits, S_1 and S_2 closely spaced, illuminated by a monochromatic source of light S. A screen is placed at a distance D from the slit to observe the interference pattern.

In the figure,

d → Slit separation

D → Slit and screen separation

$\lambda \rightarrow$ Wavelength of light

Y $\rightarrow$distance of interfering point from the centre of slit

$\Delta x \rightarrow$Path difference coming from the light S_1 and S_2

Optical path difference between the rays coming through S_1 and S_2

Now the path difference,

$$\Delta x = S_2P - S_1P$$

In figure, $S_2P = [S_2C^2 + PC^2]^{1/2}$

$$= [D^2 + (y + \frac{d}{2})^2]^{1/2}$$

$$= D[1 + \frac{(y+\frac{d}{2})^2}{D2^2}]^{1/2}$$

$$= D\left[1 + \frac{(y+\frac{d}{2})^2}{2D^2}\right] \text{ (Using binomial theorem)}$$

$$S_2P = D + \frac{(y+\frac{d}{2})^2}{2D^2}$$

Similarly,

$$S_1P = D + \frac{(y-\frac{d}{2})^2}{2D}$$

$$\Delta x = D + \frac{(y + \frac{d}{2})^2}{2D} - D - \frac{(y - d/2)^2}{2D}$$

$$= \frac{1}{2D}\left[(y + \frac{d}{2})^2 - (y - d/2)^2\right]$$

$$= \frac{1}{2D} \times 4y\frac{d}{2}$$

$$= y\frac{d}{D}$$

$$\Delta x = y\frac{d}{D}$$

The alternative dark and bright patches obtained on the interference screen due to superposition of light waves are known as fringe.

Condition for bright fringe

The bright fringe is obtained when the path difference is integral multiple of λ i.e.

$$\Delta x = n\lambda$$

From equation (4) and (5), we get

$$y_n \frac{d}{D} = n\lambda$$

$$y_n = \frac{n\lambda D}{d} \quad \text{Where n = 0, 1, 2 ……}$$

Condition for dark fringe

It will be obtained when the path difference is an odd multiple of λ/2 i.e.

$$\Delta x = \frac{(2n+1)}{2\lambda}$$

From (4) and (6), we get

$$\frac{ynd}{D} = \frac{(2n+1)\lambda}{2}$$

$$y_n = \frac{(2n+1)\lambda}{2} \quad \text{Where n = 0, 1, 2 ……}$$

Fringe Width

The separation between two consecutive dark fringes and bright fringes is known as fringe width.

If y_n and y_{n-1} be the two consecutive bright fringe.

$$\beta = y_n - y_{n-1}$$

$$= \frac{n\lambda D}{d} - \frac{(n-1)\lambda D}{d}$$

$$\beta = \frac{\lambda D}{d}$$

Similarly, is y_n and y_{n-1} be the two consecutive dark fringes.

$$\beta' = (2n+1)\frac{\lambda D}{2d} - [2(n-1)+1] \times \frac{\lambda D}{2d}$$

$$= \frac{\lambda D}{2d} + \frac{\lambda D}{2d}$$

$$\beta' = \frac{\lambda D}{d}$$

It is concluded that the separation between the two consecutive bright fringes is equal to the consecutive dark fringes.

So $\beta = \beta'$

Hence bright and dark fringes are equispaced.

Discussion:

From the expression for $\beta = \beta' = \frac{\lambda D}{d}$

$$\Rightarrow \beta \propto \lambda$$

$$\Rightarrow \beta \propto D$$

$$\Rightarrow \beta \propto \frac{1}{d}$$

If young double slit apparatus is shifted from air to any medium having refractive index (μ), fringe pattern will remain unchanged and the fringe width decreases (1/μ) as λ decreases.

$$C = f\lambda_0$$

$$\mu = \frac{C}{V} = \frac{f\lambda_0}{f\lambda_m}$$

$$\Rightarrow \lambda_m = \frac{\lambda_0}{\mu}$$

If YDSE is shifted from air to water, the fringe width decreases3/4 times width in air.

$$\beta_w = \frac{\lambda_w D}{d}$$

When YDSE is performed with white light instead of monochromatic light we observed,

I. Fringe pattern remains unchanged
II. Fringe width decreases gradually
III. Central fringe is white and others are coloured fringes overlapping

When YDSE is performed with red, blue and green light

$$\lambda_R > \lambda_G > \lambda_B$$

So $\beta_R > \beta_G > \beta_B$

$$\mu = \frac{C}{V} = \frac{f\lambda_0}{f\lambda_m}$$

$$\mu = \frac{\lambda_0}{\lambda_m}$$

$$\Rightarrow \left[\lambda_m = \frac{\lambda_0}{\mu}\right]$$

Wavelength of light in any given medium, decreases to1/μ times of wavelength in vacuum.

$$\beta \propto \lambda_m$$

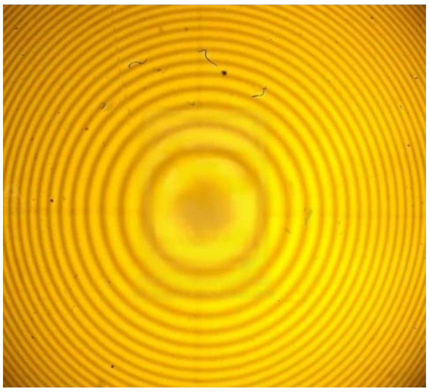
$$\beta_m = \frac{\lambda_m D}{d}$$

$$\left[\beta_m = \frac{\lambda_0 D}{\mu d}\right]$$

So, it decreases 1/μ times.

Newton’s Ring

The alternate dark and bright fringe obtained at the point of contact of a Plano convex lens with its convex side placed over a plane glass plate are known as Newton’s ring as it was first obtained by Newton.



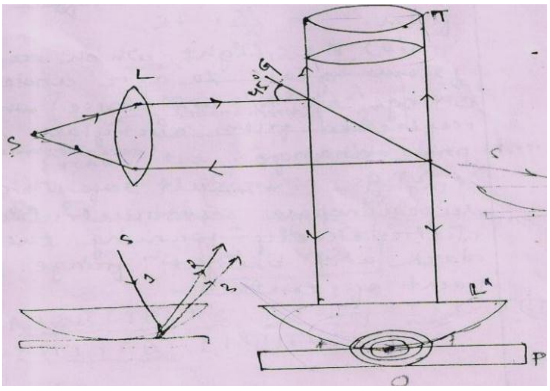
The formation of the Newton’s ring is based on the principle of interference due to division of amplitude.

Experimental Arrangement

The experimental arrangement consist of

a) S: Monochromatic source of monochromatic light

b) P: A plane glass plate



c) L: A convex lens which is placed at its focal length to make the rays parallel after refraction

d) G: A plane glass plate inclined at on 45^0 to make the parallel rays travel vertically downwards

e) L’: A plane convex lens of long focal length whose convex side kept in contact with plane glass plate

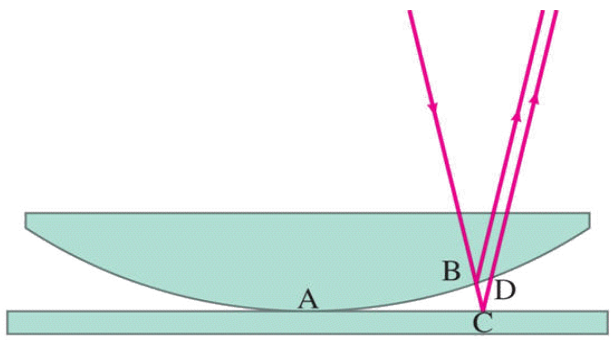
f) T: Travelling microscope mounted over the instrument to focus the Newton’s ring.

Formation of Newton’s Ring

I. To explain the formation of Newton’s ring, let us consider a plano-convex lens with its convex side kept in contact with a plane glass plate.

II. At the point of contact air film is formed whose thickness gradually goes on increasing towards outside.

III. When a beam of monochromatic light is incident on the arrangement, a part of it get reflected from the upward surface of the air film and the part of light get reflected from the lower surface of the air film.



IV. The light which reflected from glass to air undergoes a phase change of ‘π’ and those are reflected from air glass suffers no phase change.

V. As a result of which they super-impose constructively and destructively forming the alternate dark and bright fringe at the point of contact.

Condition for bright and dark fringe in Reflected light

In Newton's ring experiment, the light travels from upper and lower part of the air film suffers a path difference of λ/2 (phase change of π). Again, as the ray of light reflected twice between the air films having thickness't'. Then the total path travelled by the light is given as$(2t+\frac{\lambda}{2})$.

Now, from the condition for bright ring, we have,

$$2t+\frac{\lambda}{2}=n\lambda$$

$$2t=n\lambda-\frac{\lambda}{2}$$

$$2t=(2n-1)\frac{\lambda}{2}$$

From the condition for the dark fringe we have,

$$2t+\frac{\lambda}{2}=(2n+1)\frac{\lambda}{2}$$

$$\Rightarrow 2t=\frac{\lambda}{2}(2n+1-1)=n\lambda$$

$$\Rightarrow 2t=n\lambda$$

Newton's ring in transmitted light

The Newton's rings obtained in transmitted light are complementary to that of Newton's ring obtained in reflected light i.e.

In transmitted light, the condition for bright ring is,

$$2t = n\lambda$$

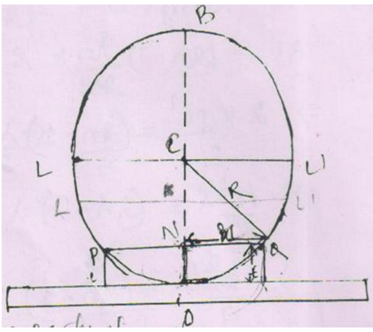
And for dark ring is,

$$2t = (2n-1)\frac{\lambda}{2}$$

Newton's ring in Reflected Light and Transmitted Light

In reflected light	**In transmitted light**
(a)Condition for bright ring; $2t = (2n-1)\frac{\lambda}{2}$ (b) Condition for bright ring; $2t = n\lambda$ (c)Newton's rings are more intense.	(a)Condition for bright ring; $2t = n\lambda$ (b) Condition for dark ring; $2t = (2n-1)\frac{\lambda}{2}$ (c)Newton's rings are less intense.

DETRMINATION OF DIAMETER OF NEWTON'S RING



LOL' is the section of lens placed on glass plate AB. C is the centre of curvature of curved surface LOL'. R is its radius of curvature and r is the radius of Newton's ring corresponding to film if thickness t.

From the property of circles,

$$PN \times NQ = ON \times NB$$

$$r \times r = t \times (2R - t)$$

t = thickness of air film

$$r^2 = 2Rt - t^2$$

$r^2 = 2Rt (\because t \ll 1)$

$$\Rightarrow t = \frac{r^2}{2R}$$

From the condition for bright Newton's ring,

$$2t = (2n - 1)\frac{\lambda}{2}$$

$$\Rightarrow 2 \times \frac{r^2}{2R} = (2n - 1)\frac{\lambda}{2}$$

$$\Rightarrow r^2 = (2n - 1)\frac{\lambda R}{2}$$

$$\Rightarrow \frac{D^2}{4} = (2n - 1)\lambda R$$

$$\Rightarrow D^2 = 2(2n - 1)\lambda R$$

$\Rightarrow D_n^2 = 2(2n - 1)\lambda R$, For the n^{th} ring.

Q) Show that diameter of Newton's dark or bright fringe is proportional to root of natural numbers.

$$D = \sqrt{2(2n - 1)\lambda R}$$

$$= \sqrt{2\lambda R \times (2n-1)}$$

$$= constant \times \sqrt{(2n-1)}$$

$\Rightarrow D_n \propto \sqrt{2n-1}$, n = 1, 2, 3…….

Thus the diameter of Newton's bright ring is proportional to square root of odd natural numbers.

Similarly from the Newton's dark ring,

$$2t = n\lambda$$

$$\Rightarrow 2 \times \frac{r^2}{2R} = n\lambda$$

$$\Rightarrow \frac{r^2}{R} = n\lambda$$

$$\Rightarrow r^2 = n\lambda R$$

$$\Rightarrow \frac{D^2}{4} = n\lambda R$$

$$\Rightarrow D^2 = 4n\lambda R$$

$$\Rightarrow D_n = \sqrt{4n\lambda R}$$

$$= \sqrt{4\lambda R}\sqrt{n}$$

$$= constant \times \sqrt{n}$$

$$D_n = constant \times \sqrt{n}$$

$$D_n \propto \sqrt{n}$$

Thus the diameter of Newton's dark ring is proportional to square root of natural numbers.

Determination of wavelength of light using Newton's ring method

To determine the wavelength of light, let us consider the arrangement which involves a travelling microscope mounted over the Newton's ring.

Apparatus, on focusing the microscope over the ring system and placing the crosswire of the eye piece on tangent position, the readings are noted. On taking readings on different positions of the crosswire on various rings we are able to calculate the wavelength of light used.

Let D_n and $D_{(n+p)}$ be the nth and (n+p)th dark ring, then we have,

$$D_n^2 = 4n\lambda R$$

$$D_{(n+p)}^2 = 4(n+p)\lambda R$$

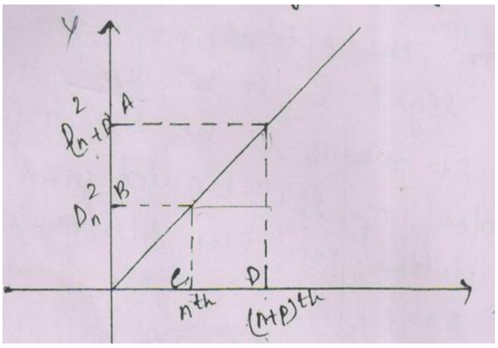
Subtracting equation (1) from (2) we get,

$$D_{(n+p)}^2 - D_n^2 = 4(n+p)\lambda R - 4n\lambda R$$

$$\frac{D_{(n+p)}^2 - D_n^2}{4PR} = \lambda$$

This is the required expression from the wavelength of light for Newton's ring method.

If we plot a graph between the orders of ring along X-axis and the diameter of the ring along Y-axis, the nature of the graph will be a straight line passing through origin.



From the graph the wavelength of light can be calculated the slope of the slope of the graph.

$\frac{1}{4R}$ Slope of the graph = wavelength of light

$$\Rightarrow \frac{AB}{CD} = \frac{D^2_{(n+p)} - D^2_n}{P}$$

$$Slope = \frac{D^2_{(n+p)} - D^2_n}{P}$$

Determination of refractive index of liquid by Newton's ring

The liquid whose refractive index is to be determined is to be placed between the gap focused between plane convex lens and plane glass plate. Now the optical path travelled by the light is to be 2μt, instead of 2t where μ be the refractive index of the liquid from the condition for the Newton's ring we have,

$$2\mu t = n\lambda$$

$$\Rightarrow 2\mu \frac{r^2}{2R} = n\lambda$$

$$\Rightarrow \mu \frac{r^2}{R} = n\lambda$$

$$\Rightarrow \frac{\mathrm{r}^2}{\mathrm{R}} = \frac{\mathrm{n}\lambda}{\mu}$$

$$\Rightarrow \frac{D^2}{4R} = \frac{\mathrm{n}\lambda}{\mu}$$

$$\Rightarrow D^2 = \frac{4n\lambda R}{\mu}$$

For n^th ring, $D^2 = \frac{4n\lambda R}{\mu}$

Let D'_{n+p} and D'_n be the diameter of the (n+p)^th and n^th dark ring in presence of liquid then

$D'^2_{n+p} = \frac{4(n+p)\lambda R}{\mu}$ and $D'^2_n = \frac{4n\lambda R}{\mu}$

Now ,

$$D'^2_{n+p} - D'^2_n = \frac{4(n+p)\lambda R}{\mu} - \frac{4n\lambda R}{\mu} = \frac{4p\lambda R}{\mu} \qquad (1)$$

If the same order ring observed in air then

$$D^2_{n+p} - D^2_n = 4p\lambda R \qquad (2)$$

Dividing equation (2) by (1) ,we have

$$\mu = \frac{\left(D^2_{n+p} - D^2_n\right)_{air}}{\left(D'^2_{n+p} - D'^2_n\right)_{liquid}}$$

This is the required expression for refractive index of the liquid.

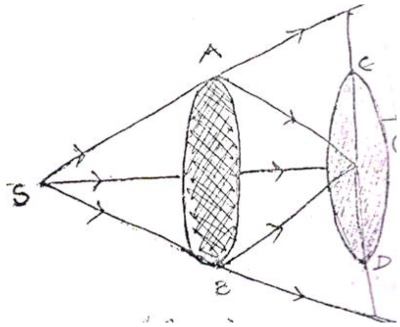
DIFFRACTION

Fundamental Idea about diffraction:

- The phenomenon of bending of light around the corner of an aperture or at the edge of an obstacle is known as diffraction
- The diffraction is possible for all types of waves
- The diffraction verifies the wave nature of light
- Diffraction takes place is due to superposition of light waves coming from two different points of a single wave front
- Diffraction takes place when the dimension of the obstacle is comparable with the wavelength of the incident light.

Explanation of diffraction:

To explain diffraction, let us consider an obstacle AB is placed on the path of an monochromatic beam of light coming from a source 'S' which produces the geometrical shadow CD on the screen. This proves the rectilinear propagation of light.



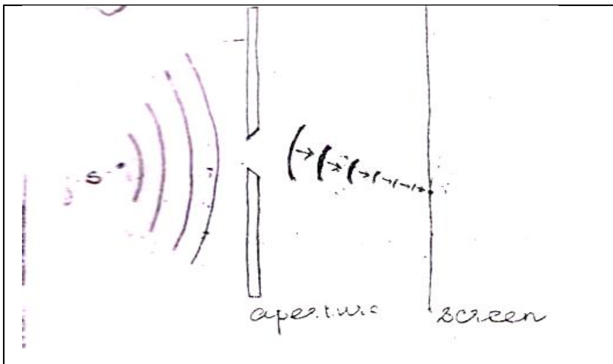
If the dimension or size of the obstacle is **comparable** with the wave length of the incident light, then light bends at the edge of the obstacle and enters in to the geometrical shadow region of the obstacle. According to Fresnel inside a well region, the destructive interference takes place for which we get brightest central maxima, which is associated with the diminishing lights on either side of the shadow as the constructive interference takes place out side the well region. This explains the diffraction phenomena.

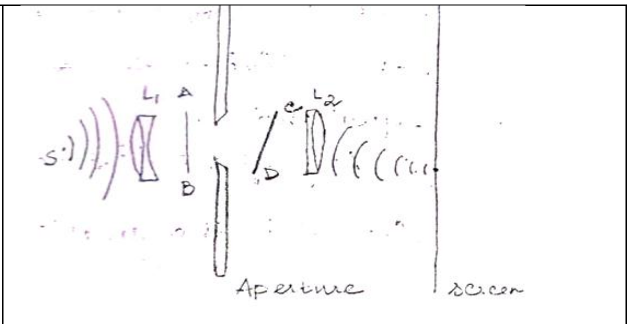
Types of Diffraction:

Depending on the relative position of the obstacle from the source and screen, the diffraction is of 2 types.

a. Fresnel Diffraction
b. Fraunhoffer Diffraction

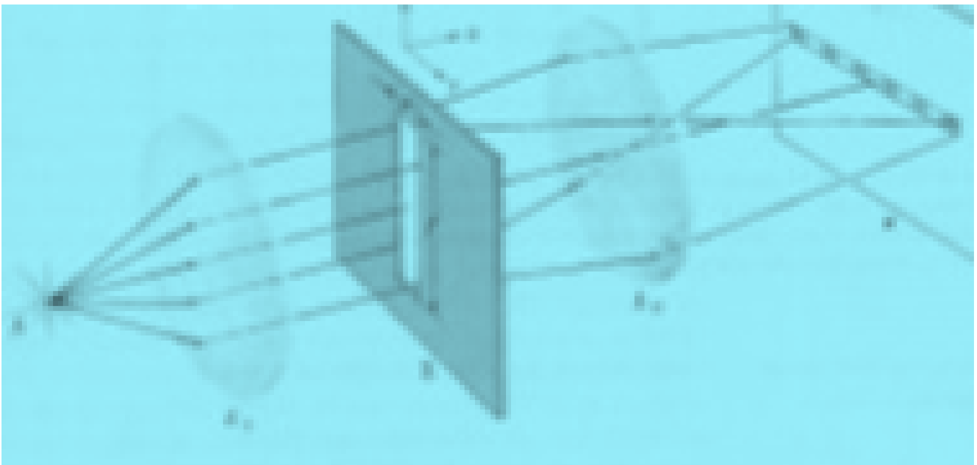
Fresnel's Diffraction	**Fraunhoffer Diffraction**
(1) The type of diffraction in which the distance of either source or screen or both from the obstacle is **finite**, such diffraction is known as Fresnel's diffraction. (2) No lenses are used to make the rays converge or parallel. (3) The incident wave front is either cylindrical or spherical. Ex:The diffraction at the straight edge.	(1) The type of diffraction in which the distance of either source or screen or both from the obstacle is **infinite**, such diffraction is known as Fraunhoffer diffraction. (2) Lenses are used to make the rays converge or parallel. (3) The incident wave front is plane. Ex:. The diffraction at the narrow.



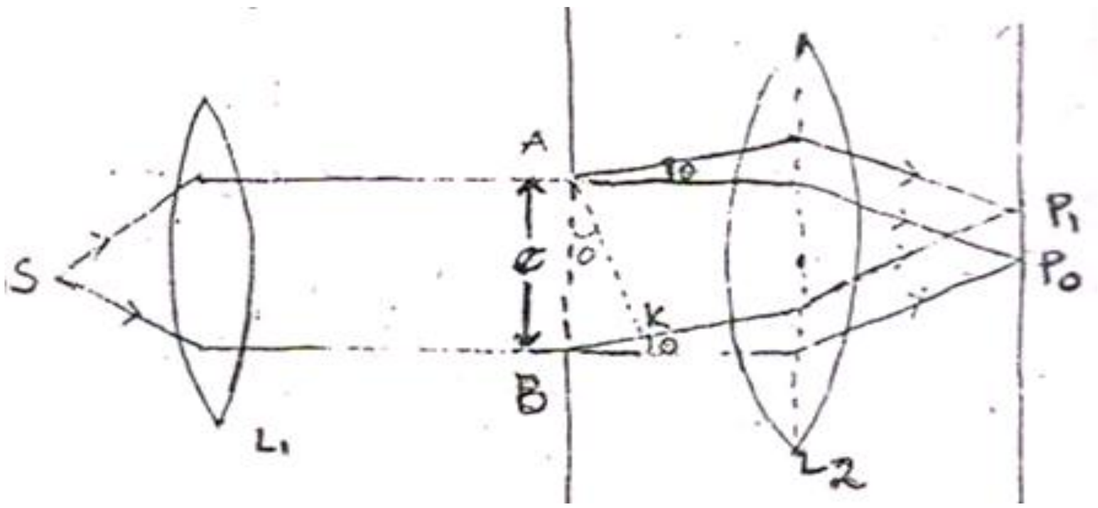


Fraunhoffer Diffraction due to a single slit:

Let us consider a parallel beam of monochromatic light inside on a slit ‘AB’ having width ‘e’. The rays of the light which are incident normally on the convex lens ‘L_2’, they are converged to a point ‘P_0’ on the screen forming a central bright image.



Fraunhoffer diffraction due to single slit



Schematic digram for Fraunhoffer diffraction due to single slit

The rays of light which get deviated by an angle ‘θ’, they are converged to a point ‘P_1’, forming an image having lens intensity.

As the rays get deviated at the slit ‘AB’ they suffer a path difference. Therefore path difference, BK = AB Sinθ

$$= e \sin\theta$$

Therefore, Phase difference = $\frac{2\pi}{\lambda} e \sin\theta$

Let us divide the single slit into ’n’ no. of equal holes and a be the amplitude of the light coming from each equal holes.

Then Avg. phase difference= $\frac{1}{n}\frac{2\pi}{\lambda} e \sin\theta$

Now the resultant amplitude due to superposition of waves is given as

$$R=\frac{a\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)}=\frac{a\sin\left\{\left(\frac{n}{2}\right)\frac{1}{n}\frac{2\pi}{\lambda}e\sin\theta\right\}}{\sin\left\{\frac{\frac{1}{n}\frac{2\pi}{\lambda}e\sin\theta}{2}\right\}}=\frac{a\sin\left(\frac{\pi}{\lambda}e\sin\theta\right)}{\sin\left(\frac{1}{n}\frac{\pi}{\lambda}e\sin\theta\right)}$$

Let $\alpha=\frac{\pi}{\lambda}e\sin\theta$,then $R=\frac{a\sin\alpha}{\sin\frac{\alpha}{n}}$

Since α is very small and n is very large so $\frac{\alpha}{n}$ is also very small.

Therefore, $\sin\frac{\alpha}{n}\approx\frac{\alpha}{n}$

Thus, R $=\frac{a\sin\alpha}{\frac{\alpha}{n}}=\frac{na\sin\alpha}{\alpha}=\frac{A\sin\alpha}{\alpha}$ $\quad where A=an$

Now the intensity is given as

Iα R^2 $\Rightarrow I=KR^2\Rightarrow I=K\frac{A^2\sin^2\alpha}{\alpha^2}=I_0\frac{\sin^2\alpha}{\alpha^2}$ where $I_0=KA$

Condition for Central /principal maxima:

When $\alpha = 0$,

$$\Rightarrow\frac{\pi}{\lambda}e\sin\theta=0\Rightarrow\sin\theta=0$$

$$\Rightarrow\theta=0$$

Thus, the condition for principal maxima will be obtained at $\theta=0$position for all the rays of light.

Position for/Condition for minima:

The minimum will be obtained when $\sin\alpha=0=\sin(\pm m\pi)$

$$\Rightarrow\sin\alpha=\sin(\pm m\pi)$$

$$\Rightarrow \alpha = \pm m\pi$$

$$\Rightarrow \frac{\pi}{\lambda} e\sin\theta = \pm m\pi$$

$\Rightarrow e\sin\theta = \pm m\lambda$ where $m = 1,2,3,4,.......$

$$\Rightarrow \theta = \pm\frac{m\lambda}{e}$$

Thus, the minimas are obtained at $\pm\frac{\lambda}{e}, \pm 2\frac{\lambda}{e}, \pm 3\frac{\lambda}{e}, \pm 4\frac{\lambda}{e}, ...$

Position/Condition for secondary maxima:

The maxima's occurring in between two consecutive secondary maxima is known as secondary maxima.

The positions for secondary maxima will be obtained as

$$\frac{dI}{d\alpha} = 0$$

$$\Rightarrow \frac{d}{d\alpha}\left[I_0 \frac{\sin^2\alpha}{\alpha^2} \right] = 0$$

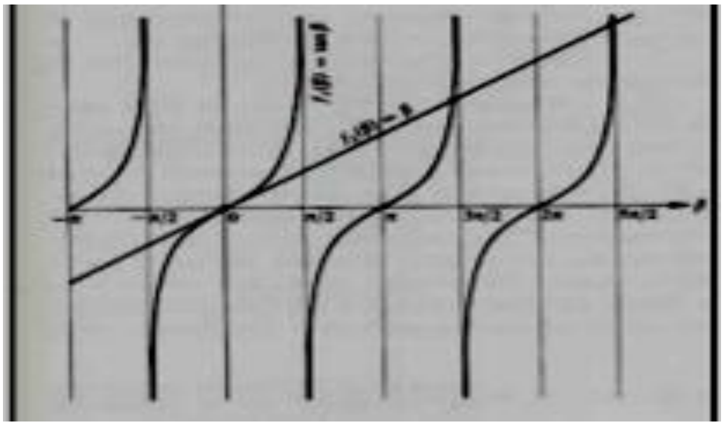
$$\Rightarrow 2I_0 \frac{\sin\alpha}{\alpha}\left[\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2} \right] = 0$$

$$\Rightarrow \frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2} = 0$$

$$\Rightarrow \alpha\cos\alpha - \sin\alpha = 0$$

$$\Rightarrow \alpha = \tan\alpha$$

This is a transdectional equation.It can be solved by graphical method. Taking $y = \alpha$ and $y = \tan\alpha$,where the two plots are interests, this intersection points gives the position for secondary maxima. Thus the secondary maxima's are obtained at $\alpha = \frac{3\pi}{2}, \alpha = \frac{5\pi}{2}, \alpha = \frac{7\pi}{2}.........$



From the expression for amplitude we have

$$R = \frac{A\sin\alpha}{\alpha} = \frac{A}{\alpha}\left[\alpha + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \frac{\alpha^7}{7!} + ...\right]$$

$$= \frac{A}{\alpha} x\alpha \left[1 + \frac{\alpha^2}{3!} + \frac{\alpha^4}{5!} + ...\right] = A, \text{ since } \alpha \ll 1$$

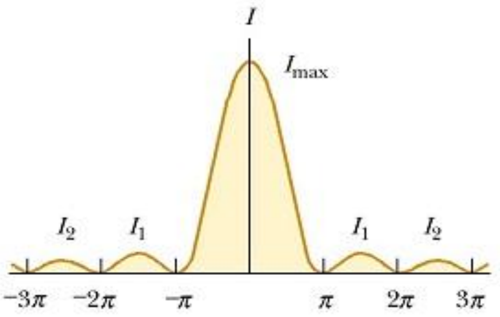
Thus the intensity at the central principal maxima is I_0

For $\alpha = \frac{3\pi}{2}$, $I_1 = I_0 \frac{Sin^2\alpha}{\alpha^2}$ = I0 $\frac{Sin^2(\frac{3\pi}{2})}{\left(\frac{3\pi}{2}\right)^2} = \frac{I_0}{22}$

For $\alpha = \frac{5\pi}{2}$, $I_2 = I_0 \frac{Sin^2\alpha}{\alpha^2}$ = I0 $\frac{Sin^2(\frac{5\pi}{2})}{\left(\frac{5\pi}{2}\right)^2} = \frac{I_0}{62}$ and so on ……

Intensity distribution curve:

The graph plotted between phase difference and intensity of the fringes is known as intensity distribution curve. The nature of the graph is as follows:



(a)

(b)

Intensity distribution curve

From the nature of the graph it is clear that

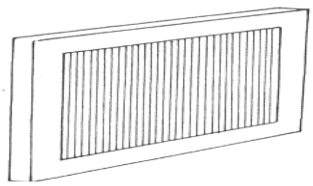
1. The graph is symmetrical about the central maximum
2. The maxima are not of equal intensity
3. The maxima are of not equal width

The minima are of not perfectly dark

PLANE TRANSMISSION GRATING:

It is an arrangement consisting of large no.of parallel slits of equal width separated by an equal opaque space is known as diffraction grating or plane transmission grating.

Diffraction grating



Construction: It can be constructed by drawing a large no. of rulings over a plane transparent material or glass plate with a fine diamond point.

Thus the space between the two lines act as slit and the opaque space will acts as obstacle.

N.B. Though the plane transmission grating and a plane glass piece looks like alike but a plane transmission grating executes rainbow colour when it exposed to sun light where as a plane glass piece does not executes so.

Grating element:

The space occurring between the midpoints of two consecutive slit in a plane transmission grating is known as **Grating element**. It can be measured by counting the no. of rulings present in a given length of grating.

Let us consider a diffraction grating having

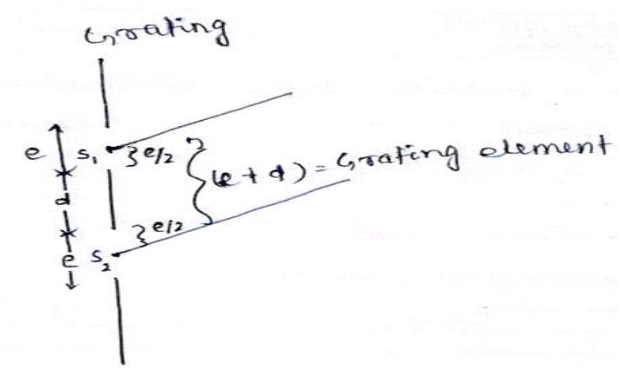
e = width of the slit

d = width of the opacity

If "N" be the no. of rulings present in a given length of grating "x" each having width (e+d), then

$$N\,(e+d) = x$$

$$\Rightarrow (e+d) = \frac{x}{N} = \text{Grating element}$$

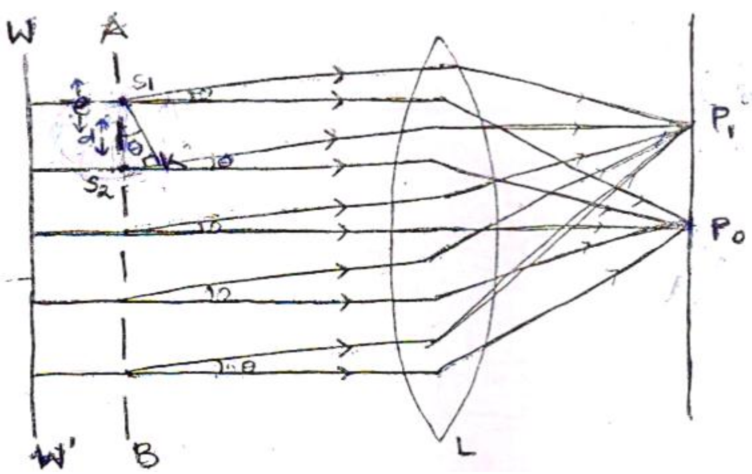


For example if a grating contain 15,000 lines per cm in a grating then the grating element of the grating

Grating element, (e+d) = $\frac{1}{15000}$ =0.00016933 cm

Diffraction due to plane transmission grating /Fraunhoffer diffraction due to N-parallel slit:

Let us consider a plane wave front coming from an infinite distance is allowed to incident on a convex lens "L" which is placed at its focal length. The rays of light which are allowed to incident normally on the lens are converged to a point “P_o” forming central principal maxima having high intensity and the rays of light which are diffracted through an angle are “θ” are converge to a point “P_1” forming a minima having less intensity as compared to central principal maxima. Again those rays of light which are diffracted through an angle “θ” are undergoes a path difference and hence a phase difference producing diffraction.



Let AB- be the transverse section of the plane transmission grating

ww'- be a plane wave front coming from infinite distance

e = width of the slit

d = width of the opacity

(e+d) = grating element of the grating

N = be the no. of rulings present in the grating

Now the path difference between the deviated light rays is

$S_2K = S_1S_2 Sin\theta = (e+d)Sin\theta$

Therefore, Phase difference = $\frac{2\pi}{\lambda}$ x $S_2K = \frac{2\pi}{\lambda}(e+d)Sin\theta = 2\beta$ (say)

where $\beta = \frac{\pi}{\lambda}(e+d)Sin\theta$

Now the resultant amplitude due to superposition of “N” no .of waves coming from “N” parallel slit is given as

$$R = A\frac{Sin\alpha}{\alpha}\frac{SinN\beta}{Sin\beta}$$

and intensity is given as

$$I\alpha R^2 \Rightarrow I = KR^2 = KA^2\frac{Sin^2\alpha}{\alpha^2}\frac{Sin^2 N\beta}{Sin^2\beta} = I_0\frac{Sin^2\alpha}{\alpha^2}\frac{Sin^2 N\beta}{Sin^2\beta}$$

where $I_0\frac{Sin^2\alpha}{\alpha^2}$ =this is contributed due to diffraction at single slit

and $\frac{Sin^2 N\beta}{Sin^2\beta}$ = this is contributed due to interference at ” N” parallel slit

Position for central principal maxima /condition for central principal maxima:

The principal maxima will be obtained when

$$Sin\beta = o = Sin(\pm m\pi)$$
$$\Rightarrow \beta = \pm m\pi$$
$$\Rightarrow \frac{\pi}{\lambda}(e+d)Sin\theta = \pm m\pi$$
$$\Rightarrow (e+d)Sin\theta = \pm m\lambda$$

where $m = 0,1,2,3......$ .This is called grating equation or condition for central principal maxima.

Position for minima /condition for minima:

The minima will be obtained when

$$SinN\beta = o = Sin(\pm n\pi)$$
$$\Rightarrow N\beta = \pm n\pi$$
$$\Rightarrow N\frac{\pi}{\lambda}(e+d)Sin\theta = \pm n\pi$$
$$\Rightarrow N(e+d)Sin\theta = \pm n\lambda$$

Where n can take all the values except $n = 0, \pm N, \pm 2N, \pm 3N, ...$

This is the condition for minima due to diffraction at N-parallel slit.

Position/Condition for secondary maxima:

The maxima's occurring in between two consecutive secondary maxima is known as secondary maxima.

The positions for secondary maxima will be obtained as

$$\frac{dI}{d\alpha} = 0$$

$$\Rightarrow \frac{d}{d\alpha}\left[I_0 \frac{Sin^2\alpha}{\alpha^2}\frac{Sin^2 N\beta}{Sin^2\beta}\right] = 0$$

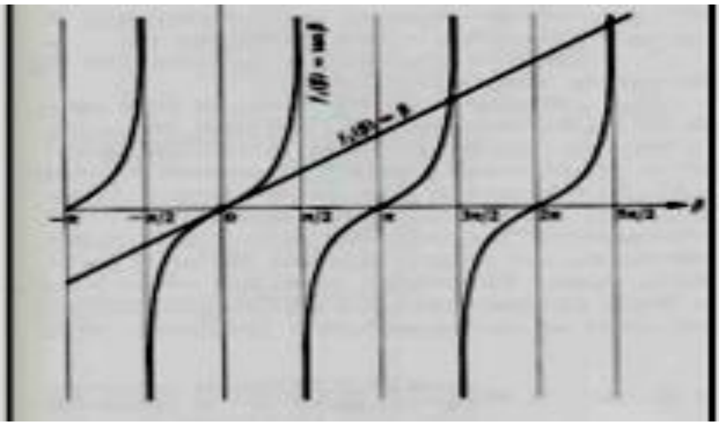
$$\Rightarrow 2I_0 \frac{Sin^2\alpha}{\alpha^2}\frac{Sin\ \ N\beta}{Sin\beta}\left[\frac{N\cos N\beta \sin\beta - \sin N\beta \cos\beta}{\sin^2\beta}\right] = 0$$

$$\Rightarrow \frac{N\cos N\beta \sin\beta - \sin N\beta \cos\beta}{\sin^2\beta} = 0$$

$$\Rightarrow N\cos N\beta \sin\beta - \sin N\beta \cos\beta = 0 \Rightarrow N\cos N\beta \sin\beta = \sin N\beta \cos\beta$$

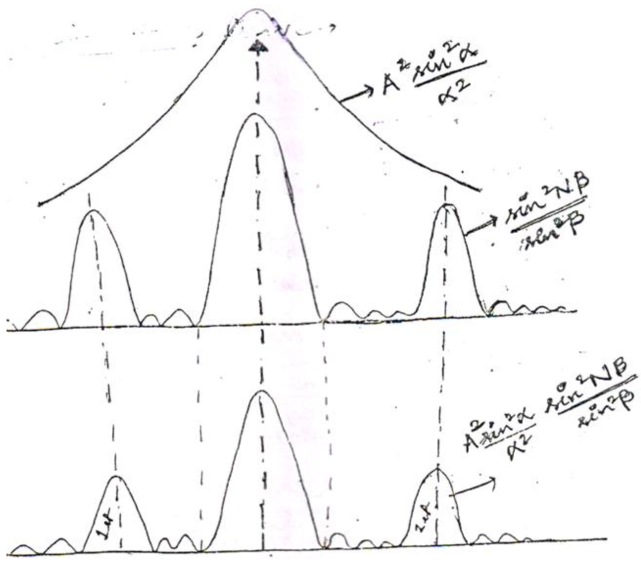
$$\Rightarrow N\tan N\beta = \tan N\beta$$

This is a transdectional equation. It can be solved by graphical method. Taking $y = \tan N\beta$ and $y = N\tan N\beta$,where the two plots are interests, this intersection points give the position for secondary maxima.Thus the secondary maxima's are obtained at $\beta = \frac{3\pi}{2}, \beta = \frac{5\pi}{2}, \beta = \frac{7\pi}{2}.........$



Intensity distribution curve:

The graph plotted between phase difference and intensity of the fringes is known as intensity distribution curve. The nature of the graph is as follows:



Characteristics of the spectral lines or grating spectra:

1.The spectra of different order are situated on either side of central principal maximum
2.Spectral lines are straight and sharp
3.The spectra lines are more dispersed as we go to the higher orders.

4.The central maxima is the brightest and the intensity decreases with the increase of the order of spectra.

Missing spectra or Absent spectra:

When the conditions for minima due to diffraction at single slit and condition for central principal maxima due to diffraction at N-parallel slit is satisfied simultaneously for a particular angle of diffraction then, certain order maxima are found to be absent or missed on the resulting diffraction pattern which are known as missing spectra or absent spectra.

Condition for Missing spectra:

We have,

The condition for central principal maxima due diffraction at N-parallel slit

$$(e+d)Sin\theta = \pm m\lambda$$

$$e\sin\theta = \pm n\lambda$$

$$\Rightarrow \frac{(e+d)Sin\theta}{e\sin\theta} = \frac{m\lambda}{n\lambda} = \frac{m}{n}$$

Special case:

1. If d = e, $\Rightarrow \frac{m}{n} = 2 \Rightarrow m = 2n$ where $n = 1,2,3,.....$

i.e second order or multiple of 2 order spectra will found to be missed or absent on the resulting diffraction pattern.

2. If $d = \frac{e}{2}, \Rightarrow \frac{m}{n} = \frac{3}{2} \Rightarrow m = 1.5n \approx 1$

i.e First order spectra will found to be missed or absent on the resulting diffraction pattern.

3. If $e = \frac{d}{2}, \Rightarrow \frac{m}{n} = 3 \Rightarrow m = 3n$

i.e Third order spectra or multiple of 3 spectra will found to be missed or absent on the resulting diffraction pattern.

Dispersion:

The phenomenon of splitting of light wave into different order of spectra is known as dispersion.

Dispersive power:

The variation of angle of diffraction with the wave length of light is known as dispersive power. It is expressed as $\frac{d\theta}{d\lambda}$

Where $d\theta = \theta_1 - \theta_2$ = difference in angle of diffraction and $d\lambda = \lambda_1 - \lambda_2$=difference in wave length of light

Expression for dispersive power:

We have

$$(e+d)Sin\theta = \pm m\lambda$$

$$\frac{d}{d\lambda}\left[(e+d)Sin\theta = \pm m\lambda\right] = \frac{d}{d\lambda}(m\lambda)$$

$$\Rightarrow (e+d)\frac{d}{d\lambda}(Sin\theta) = m\frac{d\lambda}{d\lambda}$$

$$\Rightarrow (e+d)\cos\theta\frac{d\theta}{d\lambda} = m$$

$$\Rightarrow \frac{d\theta}{d\lambda} = \frac{m}{(e+d)\cos\theta}$$

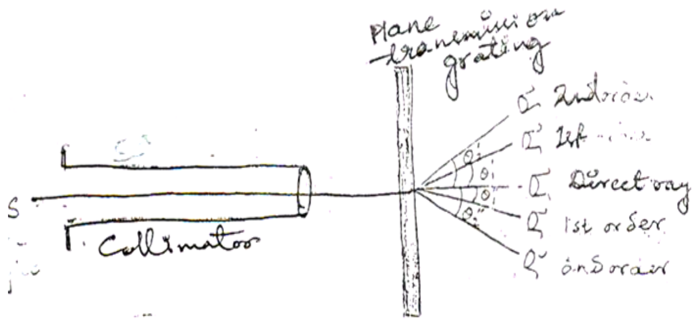
$$\Rightarrow \frac{d\theta}{d\lambda} \quad \alpha \quad m$$

$$\alpha \frac{1}{(e+d)}$$

$$\alpha \frac{1}{\cos\theta}$$

Determination of wave length of light using plane transmission grating:

To determine the wave length of light let us consider a plane transmission grating with its rulled surface facing towards the source of light perpendicular to the axis of the spectrometer. The parallel beam of monochromatic light coming from source is allowed to incident on the transmission grating which are now defracted by different angle of diffraction.Rotating the telescope for different positions of the defracted ray the angles are measured.



Using the grating equation ,

$$(e+d)Sin\theta = \pm m\lambda$$

$$\Rightarrow \lambda = \frac{(e+d)Sin\theta}{m}$$

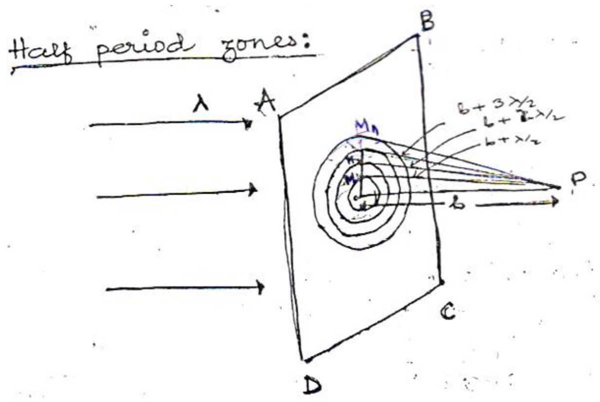
We can calculate the wave length of the monochromatic light.

Half period zone:

The space enclosed between two consecutive circles which are differing by phase of π or by a path difference of $\frac{\lambda}{2}$ or a time period of $\frac{T}{2}$ is known as half period zone. As it was first observed by Fresnel, these are also known as Fresnel half period zone.

Construction:

To construct the half period zone let us consider a plane wave front of monochromatic source of light having wavelength λ coming from left to right. Let “P” be a point just ahead of the plane wave front at a perpendicular distance “b” from the plane wavefront. Taking “P” as centre and radii equal to $OM_1 = r_1, OM_2 = r_2, OM_3 = r_3 \ldots OM_n = r_n$ let us divide the plane wave front into large no. of concentric circles such that light coming from each consecutive half period zone will differ by a phase difference of $\frac{\lambda}{2}$.



These alternative circles which are now differing by a phase change of π are known as half period zone. These half period zones are known as Fresnel half period zone. The Fresnel’s first half period zone is brighter than that of a second half period zone and the two half period zone are differ by a phase change of π.

Properties of Half period Zone:

1. **Phase of Half period Zone**: Each half period zone are differ by a phase change of π

2. **Area of half period zone**:

The space enclosed between two consecutive half period zones is called area of Half period zone.

Let A_{n-1} and A_n be the area of $(n-1)^{th}$ and n^{th} half period zone

Then, $A_{n-1} = \pi (OM_{n-1})^2 = \pi(PM_1^2 - OP^2) = \pi\left[\left(b+\frac{(n-1)\lambda}{2}\right)^2 - b^2\right]$

$$= \pi\left[b^2 + \frac{(n-1)^2\lambda^2}{4} + 2b\frac{(n-1)\lambda}{2} - b^2\right]$$

$$=\pi\left[\frac{(n-1)^2\lambda^2}{4}+b(n-1)\lambda\right]= \pi(n-1)b\lambda$$

Since $\lambda \langle\langle 1$ so $\frac{(n-1)^2\lambda^2}{4} \langle\langle 1$ and hence neglected

and $A_n = \pi\,(OM_n)^2 = \pi(PM_n^2 - OP^2) = \pi\left[\left(b+\frac{n\lambda}{2}\right)^2 - b^2\right]$

$$=\pi\left[b^2+\frac{n^2\lambda^2}{4}+2b\frac{n\lambda}{2}-b^2\right]$$

$$=\pi\left[\frac{n^2\lambda^2}{4}+bn\lambda\right]= \pi nb\lambda$$

Since $\lambda \langle\langle 1$ so $\frac{n^2\lambda^2}{4} \langle\langle 1$ and hence neglected

Now the area of the half period zone

$A = A_n - A_{n-1} = \pi nb\lambda - \pi\,(n-1)\,b\lambda = \pi b\lambda$

Thus the area of half period zone is independent of order of zone and the half period zones are equispaced

3. **Radius of half period zone**:

We have,

The area of first half period zone is $\pi b\lambda$

i.e

$A_1 = \pi b\lambda$

Again, $A_1 = \pi r_1^2$

$\Rightarrow \pi r_1^2 = \pi b\lambda \Rightarrow r_1^2 = b\lambda \Rightarrow r_1 = \sqrt{1b\lambda}$

Similarly, the radius of the second half period zone is $r_2 = \sqrt{2b\lambda}$ and the radius of the third half period zone is $r_3 = \sqrt{3b\lambda}$,……. $r_n = \sqrt{nb\lambda}$.

Thus it is found that radius of the half period zone is dependent on order of zone and the radius of the half period zone is varies directly proportional to the square root of the natural number.

Factors affecting amplitude of half period zones:

The factors affecting the amplitude are:

a. Area of half period zone (directly)

b. Average distance of half period zone (inversely)
c. Obliquity factor (directly)
Mathematically,

If 'R' be the radius of the half period zone, then

$$R \alpha A$$
$$\alpha (1+\cos\theta)$$
$$\alpha \frac{1}{d}$$

$$\Rightarrow R \alpha \frac{A(1+\cos\theta)}{d}$$

Expression for amplitude of half period zone:

Let $R_1, R_2, R_3 \ldots\ldots R_n$ be the amplitudes of 1st, 2nd, 3rd,……nth half period zone respectively.

Then the net amplitude due to the entire half period zone is given by

$$R = R_1 + R_2 + R_3 + \ldots\ldots\ldots R_n$$
$$= R_1 - R_2 + R_3 - \ldots\ldots\ldots + .R_n \quad \text{(If n is odd)}$$
$$= R_1 - R_2 + R_3 - \ldots\ldots\ldots + .R_{n-1} \quad \text{(If n is even)}$$

Since $R_1 \rangle R_2, R_2 \rangle R_3$ so we have

$R_2 = \frac{R_1 + R_3}{2}$ and $R_4 = \frac{R_3 + R_5}{2}$ and so on

$$R = \left(\frac{R_1}{2} - R_2 + \frac{R_3}{2}\right) + \left(\frac{R_1}{2} - R_4 + \frac{R_5}{2}\right) + \ldots\ldots\ldots \left(\frac{R_n}{2}\right) \text{ if n is even}$$

$$= \left(\frac{R_1}{2} - R_2 + \frac{R_3}{2}\right) + \left(\frac{R_1}{2} - R_4 + \frac{R_5}{2}\right) + \ldots\ldots\ldots \left(\frac{R_{n-1}}{2}\right) \text{ if n is odd}$$

$$R = \frac{R_1}{2} + \frac{R_n}{2} \text{ if n is odd}$$

$$= \frac{R_1}{2} + \frac{R_{n-1}}{2} \text{ if n is even}$$

As n $\rangle\rangle\rangle 1$ and $\frac{R_{n-1}}{2} \, or \, \frac{R_n}{2} \langle\langle\langle 1$ so, $R = \frac{R_1}{2}$

Thus the net amplitude due to entire half period zone is equal to half of the amplitude due to first half period zone.

Zone plate:

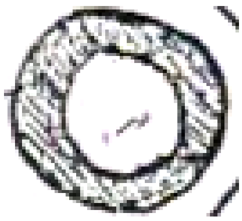
A special diffracting screen which obstructs the light from alternate half period zone is known as zone plate.

Construction:

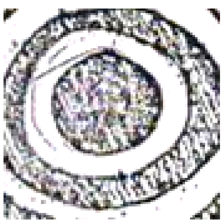
It can be constructed by drawing a series of concentric circles on a white sheet of paper with radii proportional to square root of natural number. The alternate half period zones are painted black. A reduced photograph of this drawing is taken on a plane glass plate. The negative thus obtained act as zone plate.

Depending on the initial blackening the zone plate is of two types

1. Positive zone plate: zone plate: the the center is bright
2. Negative

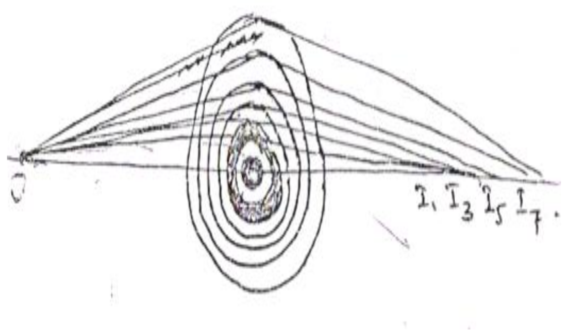


center is dark



Working:

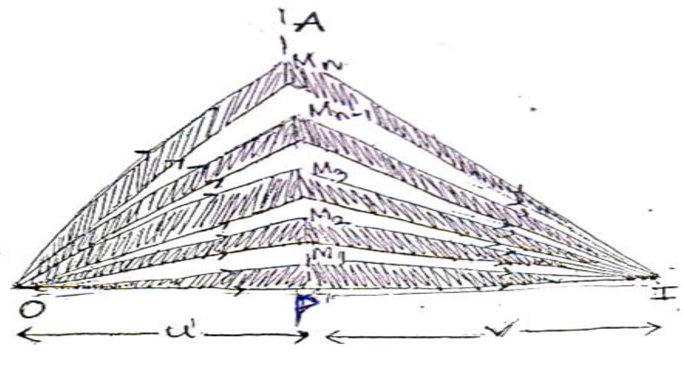
When a beam of monochromatic light is allowed to fall on a zone plate, the light is obstructed from the alternate half period zone through the alternate transparent zones. So,the rays of light differ by a phase difference of π.



Hence, the resultant amplitude is sum of the individual amplitude due to light coming from alternate half period zones. Thus for any point object situated at infinite produces a bright image at a particular distance which is same as that of image produced by a convex lens. Thus a zone plate is equivalent to that of a convex lens.

Theory of zone plate:

Let us consider a transverse section of a zone plate placed perpendicular to the plane of the paper. Let ‘O’ be a point object placed at a distance ‘$OP = u$’ forms a real image ‘I’ at a distance ‘$PI = v$’ from the zone plate.



Taking 'P' as center and radii equal to $PM_1 = r_1, PM_2 = r_2, PM_3 = r_3 \ldots PM_n = r_n$, the entire plane of the paper is divided into large no. of concentric circles such that the light coming from alternate half period zone will differ by a path difference of $\frac{\lambda}{2}$ in such a way that

$$OM_1I - OPI = \frac{\lambda}{2}$$

$$OM_2I - OPI = \frac{2\lambda}{2}$$

$$OM_3I - OPI = \frac{3\lambda}{2}$$

……………………..

……………………..

$$OM_nI - OPI = \frac{n\lambda}{2}$$

$$OM_n + I - PM_nI = \frac{n\lambda}{2} \qquad (1)$$

Now, in right angled ΔOPM_n

$$OM_n = \left[OP^2 + PM^2{}_n\right]^{\frac{1}{2}}$$

$$= \left[u^2 + r^2{}_n\right]^{\frac{1}{2}} = u\left[1 + \frac{r^2{}_n}{u^2}\right]^{\frac{1}{2}} = u\left[1 + \frac{r^2{}_n}{2u^2} + \ldots..\right], \qquad \text{as } r_n \langle u$$

$$= u\left[1 + \frac{r^2{}_n}{2u^2}\right] = u + \frac{r^2{}_n}{2u} \qquad (2)$$

Similarly, in right angled ΔPM_nI

$$M_nI = \left[M_nI^2 + PM^2{}_n\right]^{\frac{1}{2}}$$

$$= \left[v^2 + r^2{}_n\right]^{\frac{1}{2}} = v\left[1 + \frac{r^2{}_n}{v^2}\right]^{\frac{1}{2}} = v\left[1 + \frac{r^2{}_n}{2v^2} + ...\right] , \qquad \text{as } r_n \langle v$$

$$= v\left[1 + \frac{r^2{}_n}{2v^2}\right] = v + \frac{r^2{}_n}{2v} \qquad (3)$$

Using eq^n (2) and eq^n (3) in eq^n (1) we get

$$u + \frac{r^2{}_n}{2u} + \ v + \frac{r^2{}_n}{2v} - (u+v) = \frac{n\lambda}{2}$$

$$\Rightarrow (u+v) + \left[\frac{r^2{}_n}{2}\left(\frac{1}{u} + \frac{1}{v}\right)\right] - (u+v) = \frac{n\lambda}{2}$$

$$\Rightarrow \frac{r^2{}_n}{2}\left(\frac{1}{u} + \frac{1}{v}\right) = \frac{n\lambda}{2}$$

$$\Rightarrow r^2{}_n\left(\frac{1}{u} + \frac{1}{v}\right) = n\lambda \qquad (4)$$

$$\Rightarrow r^2{}_n\left[\frac{u+v}{uv}\right] = n\lambda$$

$$\Rightarrow r_n = \sqrt{\frac{u+v}{uv}}\sqrt{n}$$

$$\Rightarrow r_n = cons.\sqrt{n}$$

$$\Rightarrow r_n \alpha \sqrt{n}$$

Thus the radius of the zone plate is proportional to square to natural number.

Expression for primary focal length:

From eq^n. (4) we have,

$$r^2{}_n\left(\frac{1}{u} + \frac{1}{v}\right) = n\lambda$$

According to sign convention,

$$\Rightarrow r^2{}_n\left(-\frac{1}{u}+\frac{1}{v}\right)= n\lambda$$

$$\Rightarrow r^2{}_n\left(\frac{1}{f}\right)= n\lambda \quad \Rightarrow f=\frac{r_n^2}{n\lambda} \qquad (5)$$

This is the required expression for primary focal length

Again, $f\alpha\frac{1}{\lambda} \quad \Rightarrow f x\lambda$ = constant

Area of zone plate:

The space enclosed between two consecutive zones is known as area of zone plate.

Let A_{n-1} and A_n be the area of (n-1) th and nth zone

Then A= A_n - A_{n-1} = $\pi r_n^2 - \pi r_{n-1}^2 = \frac{\pi uvn\lambda}{u+v} - \frac{\pi uv(n-1)\lambda}{u+v} = \frac{\pi uv\lambda}{u+v}$ = constat

Thus, the area of zone plate is independent of order of zone i.e the zones are equispaced.

Multiple foci of zone plate:

Now from the expression we have,

$$r^2{}_n\left(\frac{1}{u}+\frac{1}{v}\right)= n\lambda$$

If the object is situated at infinity (∞), then the first image at distance ,

$$v_1 = f_1 = \frac{r_n^2}{n\lambda}$$

If we divide the half period zones into half period elements having equal area, then the 1st half period zone will divided into three half period zones,2nd half period zone will divided into five half period elements and so on

The second brightest image will at

$v_3 = f_3 = \frac{1}{3} f_1 = \frac{1}{3}\frac{r_n^2}{n\lambda}$

The third brightest image will at $v_5 = f_5 = \frac{1}{5} f_1 = \frac{1}{5}\frac{r_n^2}{n\lambda}$ and so on….

Thus it is conclude that the zone plate has multiple foci.

Comparison between the zone plate and the convex lens:

Similarities

1. Both form the real image.
2. The relation between consecutive distances is same for both.
3. In both the cases focal length depends on wave length of the light.

Dissimilarities

Convex Lens	Zone Plate
a) Image is formed by refraction b) It has a single focus. c) The focal length increases with increase of wave length. d) Image is more intense e) The optical path is constant for all the rays of light.	a) Image is formed by diffraction b) It has multiple foci c) The focal length decreases with increase of wavelength d) Image is less intense e) The optical path is different for different rays of light

Phase reversal Zone Plate:

The zone plate which is constructed in such a way that the light coming from two successive zones differ by an additional path difference of λ/2, such zone plate is known as phase reverse zone plate.

Huygens's Principle:

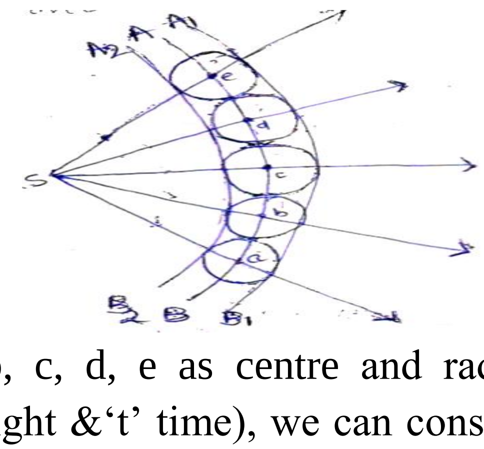
About the propagation of the wave, Huygens suggested a theory which is based on a principle known as Huygens's principle.

It states that:-

1) Each point on a given wave front will act as centre of disturbances and emits small wavelets called secondary wave front in all the possible direction.
2) The forward tangent envelope to these wave lets gives the direction of new wave front.

Explanation/construction of secondary wave front:

To explain Huygens's principles let us consider a source of light emits waves in all directions. Let AB be the wave front at t=0. As the time advances each point on the given wave front AB will act as centre of disturbance and emit wave lets in all possible directions.



Taking a, b, c, d, e as centre and radii equal to 'ct' (c-velocity of light &'t' time), we can construct a large number of spheres which represents a centre of disturbance for the new wave. The length A_1B_1 represents the direction of new wave front.

N.B. The backward front is not visible as the intensity of the backward wave front is very small since for the backward wave front,

$I = k\ (1+\cos\theta)$ since for backward wave front ($\theta=180^0$)

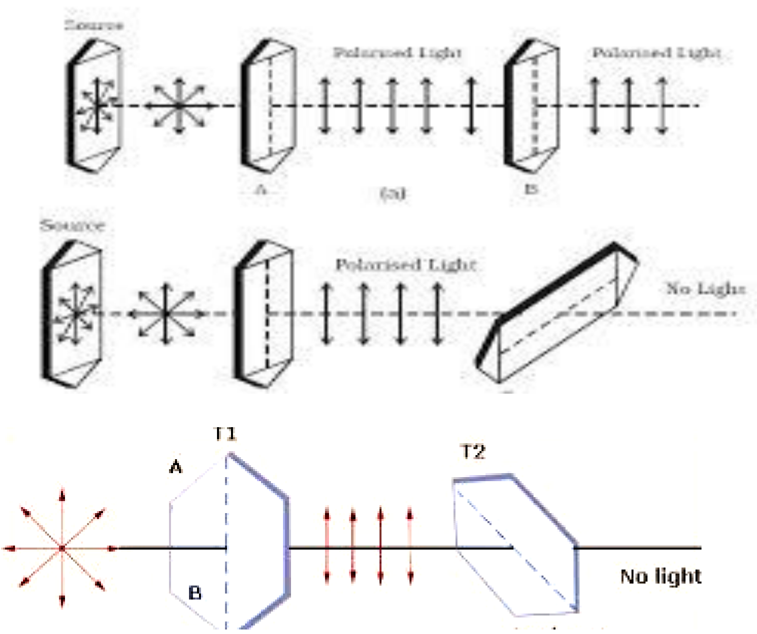
$I=k\ (1-1) = 0$

$I_{back}=0$

POLARISATION

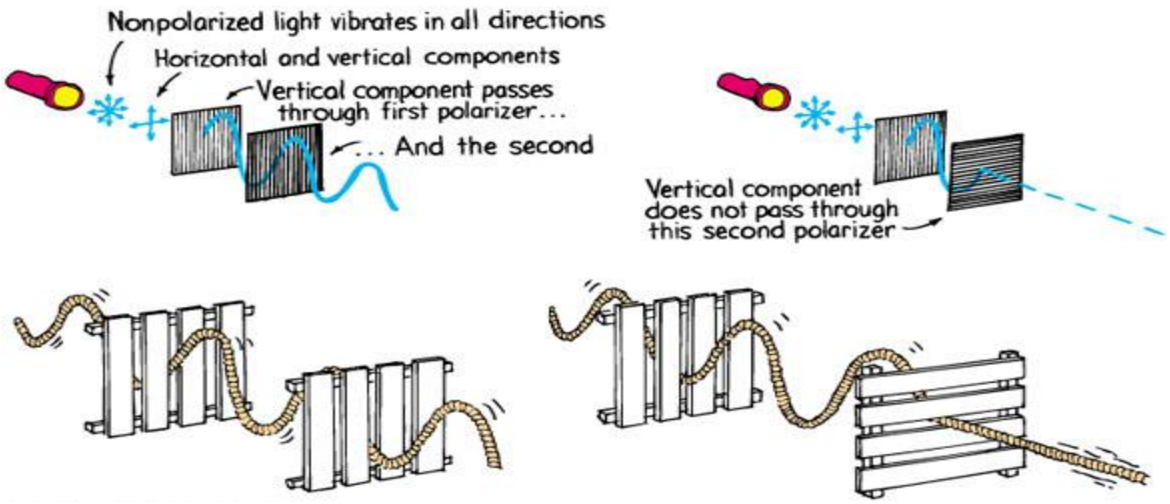
The phenomenon of restricting the vibration of light in a particular direction perpendicular to the direction of wave motion is called as polarisation.

To explain the phenomenon of polarisation let us consider the two tourmaline crystal with their optics axis placed parallel to each other .When an ordinary light is incident normally on the two crystal plates the emergence light shows a variation in intensity as T_2 is rotated.



The intensity is maximum when the axis of T_2 is parallel to that of T_1 and minimum when they are at right angle. This shows that the light emerging from T_1 is not symmetrical about the direction of propagation of light but its vibration are confined only to a single line in a plane perpendicular to the direction of propagation, such light is called as polarised light.

Example:



Difference between Polarised and ordinary light:

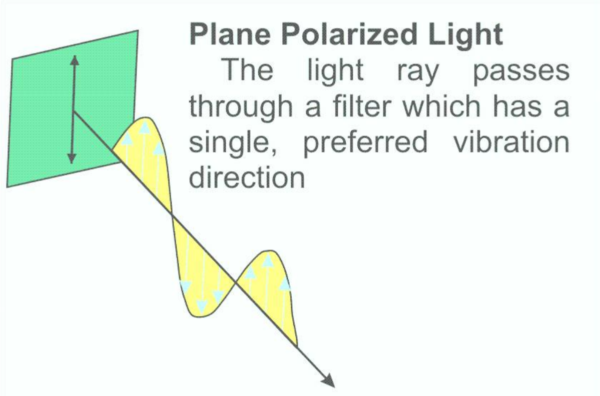
Polarised light	Ordinary light
1. The vibrations are confined in a particular direction. 2. The probability of occurrence of vibration along the axis of crystal is not same in all position of crystal 3. The intensity of light plate is not same in all position of the crystal plate.	1. The vibrations of light particle are not confined in a particular direction. 2. The probability of occurrence of vibration along the axis of the crystal is not symmetries for all position of the crystal. 3. The intensity of light plate is same in all position of the plate.

Polarised light:

The resultant light wave in which the vibrations are confined in a particular direction of propagation of light wave, such light waves are called Polarised light. Depending on the mode of vibration in a particular direction, the polarised light is three types

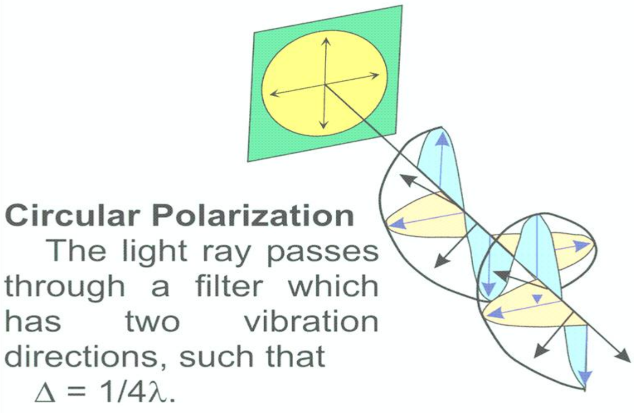
Linearly Polarised /Plane polarised:

When the vibrations are confined to a single linear direction at right angles to the direction of propagation, such light is called Plane polarised light.



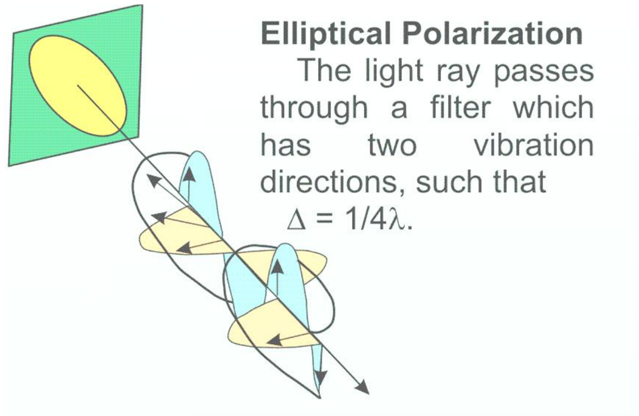
Circularly polarised light:

When the two plane polarised wave superpose under certain condition such that the resultant light vector rotate with a constant magnitude in a plane perpendicular to the direction of propagation and tip of light vector traces a circle around a fixed point such light is called circularly polarised light.



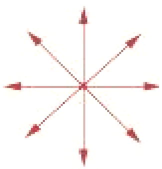
Elliptically polarised light:

When two plane polarised light are superpose in such a way that the magnitude of the resultant light vector varies periodically during its rotation then the tip of the vector traces an ellipse such light is called elliptically polarised light.

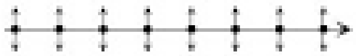


Pictorial representation of polarised light:

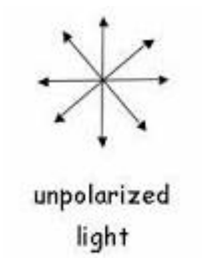
Since in unpolarised light all the direction of vibration at right angles to that of propagation of light. Hence it is represented by star symbol.



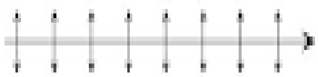
Ordinary light



non-Polarized



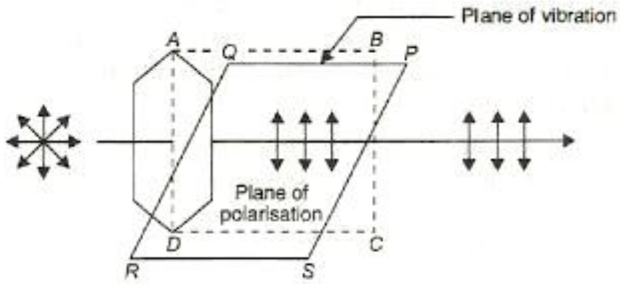
In a plane polarised beam of light, the polarisation is along straight line, the vibration are parallel to the plane and can be represented by



If the light particles vibrate along the straight line perpendicular to the plane of paper, then they can be represented by a dot.

Plane of vibration:

The plane containing the direction of vibration and direction of propagation of light is called as plane of vibration.



Plane of polarisation:

The plane passing through the direction of propagation and containing no vibration is called as plane of polarisation.

Since a vibration has no component of right angle, to its own direction, so the plane of polarisation is always perpendicular to the plane of vibrations.

Angle between plane of vibration and plane of polarisation is 90˚.

Light waves are transverse in nature:

If the light waves are longitudinal in nature, they will show no variation of intensity during the rotation of the crystal. Since during the rotation of the crystal, the variation in intensity takes place, this suggests that light waves are transverse in nature rather longitudinal.

Production of polarised light:

The polarised light can be produced in four different ways such as

1. Polarisation by Reflection

2. Polarisation by Refraction

3. Polarisation by Scattering

4. Polarisation by Double refraction

1. Polarisation by reflection:

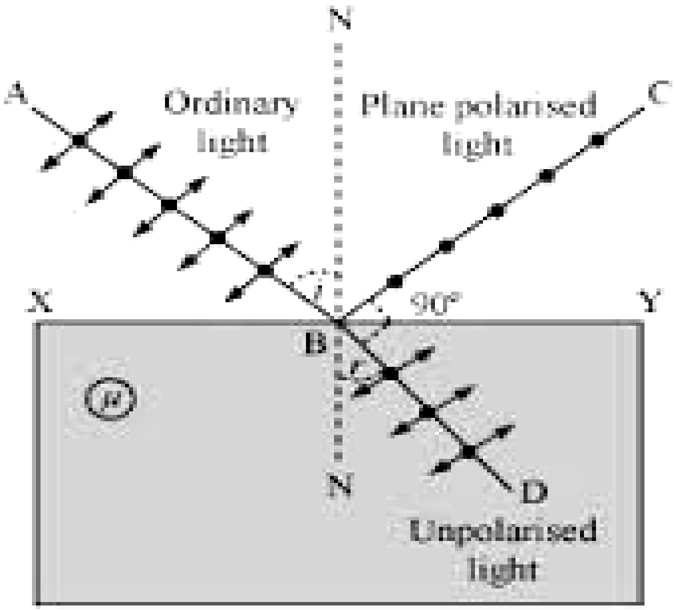
The production of the polarised light by the method of reflection from reflecting interface is called polarisation by reflection.

When the unpolarised light incident on a surface, the reflected light may be completely polarised, partially polarised or unpolarised depending upon the angle of incidence. If the angle of incidence is 0° or 90° the light is not polarised. If the angle of incidence lies in between 0° and 90°, the light is completely plane polarised.

The angle of incidence for which the reflected component of light is completely plane polarised, such angle of incidence is called polarising angle or angle of polarisation or Brewster's angle .It is denoted by i_p.

At i_p the angle between reflected ray and refracted or transmitted ray is $\pi/2$.

Explanation: To explain the polarisation by reflection, let us consider an interface XY on which a ray AB which is unpolarised is incident at an angle equal to polarising angle and get reflected along BC which is completely plane polarised and the ray BD which is refracted or transmitted is continues to be unpolarised. The incident ulpolarised light contain both perpendicular and parallel component of light.



The parallel component of light is converted into perpendicular component and gets reflected from the interface. The parallel component of light is continues to vibrate and get refracted or transmitted. As a result of which the reflected component is polarised.

Conclusion:

Hence, the reflected ray of light contains the vibrations of electric vector perpendicular to the plane incidence. Thus the reflected light is completely plane polarised perpendicular to plane of incidence.

Brewster's Law:

This law states that when an unpolarised light is incident at polarizing angle 'i_p' on an interface separating air from a medium of refractive index "μ" then the reflected light is fully polarized. i.e. $\mu = \tan i_p$

To explain Brewster's law, let XY be a reflecting surface on which;

AB = unpolarised incident light

BC= completely polarized

BD = partially polarized

i_p =angle of incidence, angle of polarization

From fig.

$\angle$CBY+$\angle$DBY=90°

$$\left(90^0 - r^{'}\right) + \left(90^0 - r\right) = 90^0$$

$$\Rightarrow \left(90^0 - i_p\right) + \left(90^0 - r\right) = 90^0$$

$$\Rightarrow \; i_p + r = 90^0$$

$$\Rightarrow r^{'} = 90^0 - r$$

From Snell's law

$$\mu = \frac{\sin i_p}{\sin r} = \frac{\sin i_p}{\sin(90^0 - i_p)} = \frac{\sin i_p}{\cos i_p} = \tan i_p$$

Thus the tangent of the angle of polarization is numerically equal to the refractive index of the medium.

NOTE: We can also prove in case of reflection at Brewster's angle reflected and refracted ray are mutually perpendicular to each other.

From Brewster's law;

We have $\mu = \tan i_p = \dfrac{\sin i_p}{\cos i_p}$

According to Snell's law;

$$\mu = \frac{\sin i_p}{\sin r}$$

From above equations

$$\sin r = \cos i_p \Rightarrow \sin r = \sin(90^0 - i_p) \Rightarrow r = 90^0 - i_p \Rightarrow r + i_p = 90^0$$

$$\Rightarrow 90^0 - \angle CBY + 90^0 - \angle DBY = 90^0$$

$$\Rightarrow \qquad \angle \text{CBY} + \angle \text{DBY} = 90^\circ$$

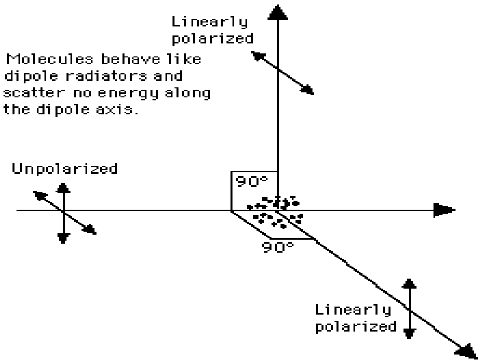
$$\Rightarrow C\vec{B} \perp B\vec{D} \Rightarrow \angle CBD = 90^0$$

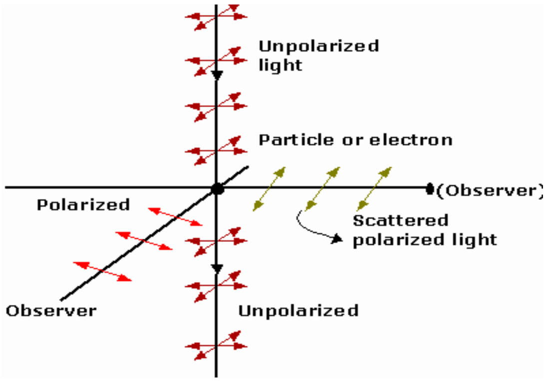
Thus, it is concluded that at polarizing angle or at Brewster's angle, the reflected light and the refracted light are mutually perpendicular to each other.

2. Polarisation by Scattering:

When a beam of ordinary light is passed through a medium containing particles, whose size is of order of wavelength of the incident light, then the beam of light get scattered in which the light particles are found to vibrate in one particular direction . This phenomenon is called "Polarisation by scattering".

Explanation:

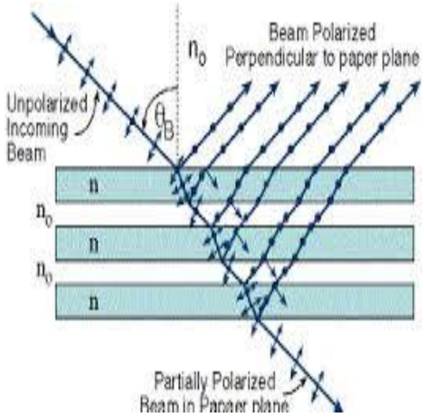


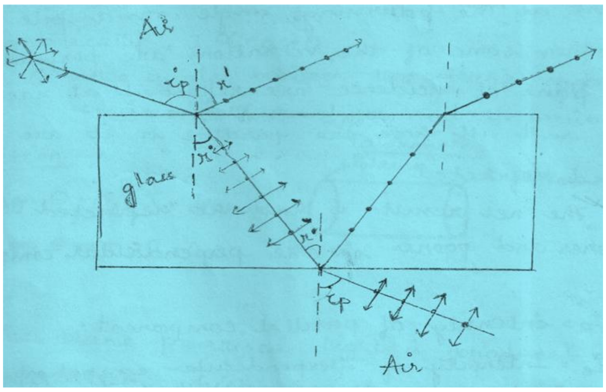


To explain the phenomenon of scattering, let us consider a beam of unpolarised light along z-axis on a scatter at origin. As light waves are transverse in nature in all possible direction of vibration of unpolarised light is confined to X-Y plane. When we look along X-axis we can see the vibrations which are parallel to Y-axis. Similarly, when we look along Y-axis the vibration along X-axis can be seen. Hence, the light can be scattered perpendicular to incident light is always plane polarized.

Polarisation by refraction:

The phenomenon of production of polarised light by the method of refraction is known as polarisation by refraction.





To explain the polarization by refraction, let us consider an ordinary light which is incident upon the upper surface of the glass slab at an polarizing angle i_p or Brewster's angle θ_B, so that the reflected light is completely polarized while the rest is refracted and partially polarized. The refracted light is incident at the lower face at an angle 'r'.

Now,

$$\tan r = \frac{\sin r}{\cos r} = \frac{\sin r}{\sin(90^0 - r)} = \frac{\sin r}{\sin i_p} = {}^g\mu_a \qquad \Rightarrow \tan r = {}^g\mu_a$$

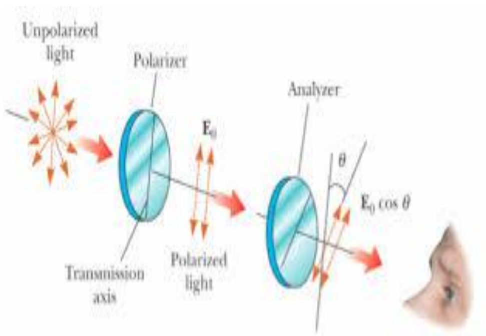
Thus according to Brewster's law, 'r' is the polarizing angle for the reflection at the lower surface of the plate. Hence, the light reflected at the lower surface is completely plane-polarised, while that transmitted part is partially polarised. Hence, if a beam of unpolarised light be incident at the polarizing angle on a pile of plates, then some of the vibrations are perpendicular to the plane of incidence are reflected at each surface and all those parallel to it are refracted. The net result is that the refracted beams are poorer and poorer in the perpendicular component and less partially polarised component.

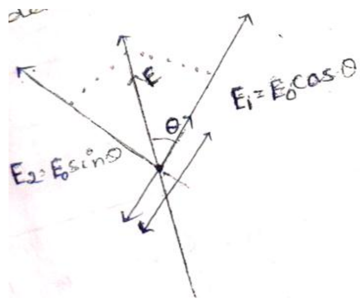
Malus law:

It states that when a beam of completely plane polarized light incident on the plane of analyser, the intensity of the transmitted light varies directly proportional to the square of the cosine of the angle between the planes of the polariser and plane of the analyser.

Mathematically,

$$I \alpha \cos^2\theta$$





Proof:

Let us consider a beam of plane polarised light coming from the plane of the polariser is incident at an angle 'θ'on the plane of the analyser. The amplitude of the light vector 'E' is now resolved into two mutual perpendicular component i.e. $E_1 = E_0 \cos\theta$ which is parallel to the plane of transmission and $E_2 = E_0 \sin\theta$ which is perpendicular to the plane of transmission. As we are able to see only the parallel component so the intensity of the transmitted light coming from the plane of the analyser is proportional to the parallel component only.

Thus,

$$I \alpha E_1^{\,2} \Rightarrow I = kE_0^2 \cos^2\theta = I_0 \cos^2\theta \text{ , where } I_0 = kE_0^2$$

$$I \alpha \cos^2\theta$$

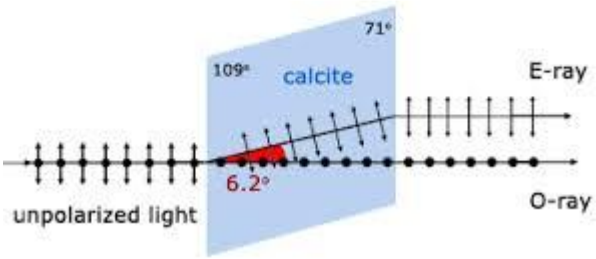
Which is Mauls law

Double refraction:

The phenomenon of splitting of ordinary light into two refracted ray namely ordinary and extra ordinary ray on passing through a double refracting crystal is known as double refraction

Explanation:

To explain the double refraction, let us consider an ordinary light incident upon section of a doubly refracting crystal



When the light passing through the crystal along the optic axis then at the optic axis the ray splits up into two rays called as ordinary and extraordinary ray which get emerge parallel from the opposite face of the crystal through which are relatively displaced by a distance proportional to the thickness of the crystal. This phenomenon is called as double refraction.

Difference between the Ordinary (O-ray)and Extra ordinary ray(E-ray)

Ordinary ray	Extraordinary ray

1.These ray obeys the law of refraction 2.For ordinary ray plane of vibration lies perpendicular to the direction of propagation 3.The vibration of particles are perpendicular to the direction of ray. 4. Plane of polarisation lies in the principal plane. 5. Refractive index is constant along optics axis. 6. It travels with the constant speed in all direction.	1. These ray do not obey law of refraction 2.For extraordinary ray the plane of vibration parallel to the direction of propagation 3. The vibration of particle is parallel to the direction of ray. 4. Plane of polarisation is perpendicular to its principal axis. 5. Refractive index varies along optics axis. 6.It travels with different speed in different direction .But it travel with equal speed along optics axis

Double refracting crystal:

The crystal which splits a ray of light incident on it into two refracted rays such crystal are called double refracting crystal.It is of two types

1. Uniaxial
2. Biaxial.

Uniaxial: The double refracting crystal which have one optic axis along which the two refracted rays travel with same velocity are known as uniaxial crystal

Ex: Calcite crystal, tourmaline crystal, quartz

Biaxial: The double refracting crystal which have two optic axis are called as biaxial crystal

Ex: Topaz, Agromite

Optic axis: It is a direction inside a double refracting crystal along which both the refracted behave like in all respect.

Principal section: A plane passing through the optic axis and normal to a crystal surface is called a principal section

Principal plane:

The plane in the crystal drawn through the optic axis and ordinary ray or drawn through the optic axis and the extraordinary ray is called as principal plane these are two principal plane corresponding to refracted ray

Polarisation by double refraction:

To explain polarisation by double refraction let us consider a beam of light incident normally through a pair of calcite crystal and rotating the second crystal about the incident ray as axis we have the following situations as:

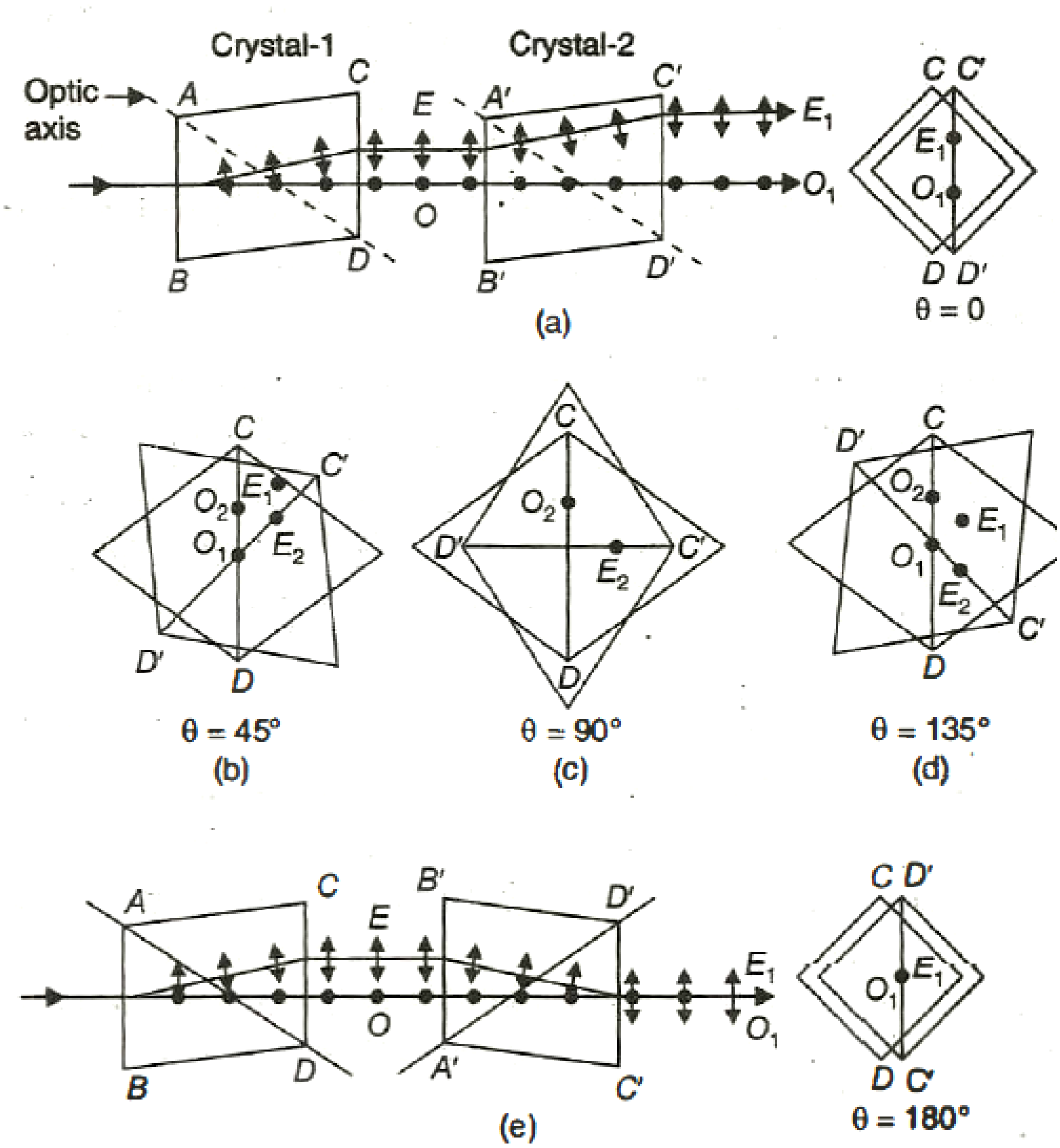
Case 1

When principal sections of two crystals are parallel then two images O_1 and E_1 are seen.

- ❖ The ordinary ray from the first crystal passes undeviated through the 2^{nd} crystal and emerges as O_1 ray. The extraordinary ray (E-ray) from the 1st crystal passes through the 2^{nd} crystal along a path parallel to that inside the 1^{st} and emerges as E_1 -ray. Hence the image O_1 and E_1 are seen separately.
- ❖ When the 2^{nd} crystal is rotated through an angle 45° with respect to 1^{st} , then the two new images O_2 and E_2 appear .As the rotation is continued , O_1 and O_2 remained fixed while E_1 and E_2 rotate around O_1 and O_2 respectively and images are found to be equal intensities.
- ❖ When the 2^{nd} crystal is rotated at an angle 90° w.r.t 1^{st} the original images O_1 and E_1 have to vanish and all the new images O_2 and E_2 have acquired the maximum intensity.

- When the 2nd crystal is rotated at an angle 135° w.r.t the 1st , four images once again appear with equally intense.

- When the 2nd crystal is rotated at an angle 180° w.r.t 1st, the O_2 and E_2 vanishes and O_1 and E_1 have come together in the centre.



This is how we are able to produce the plane polarised light by the method of double refraction.

Nicol Prism:

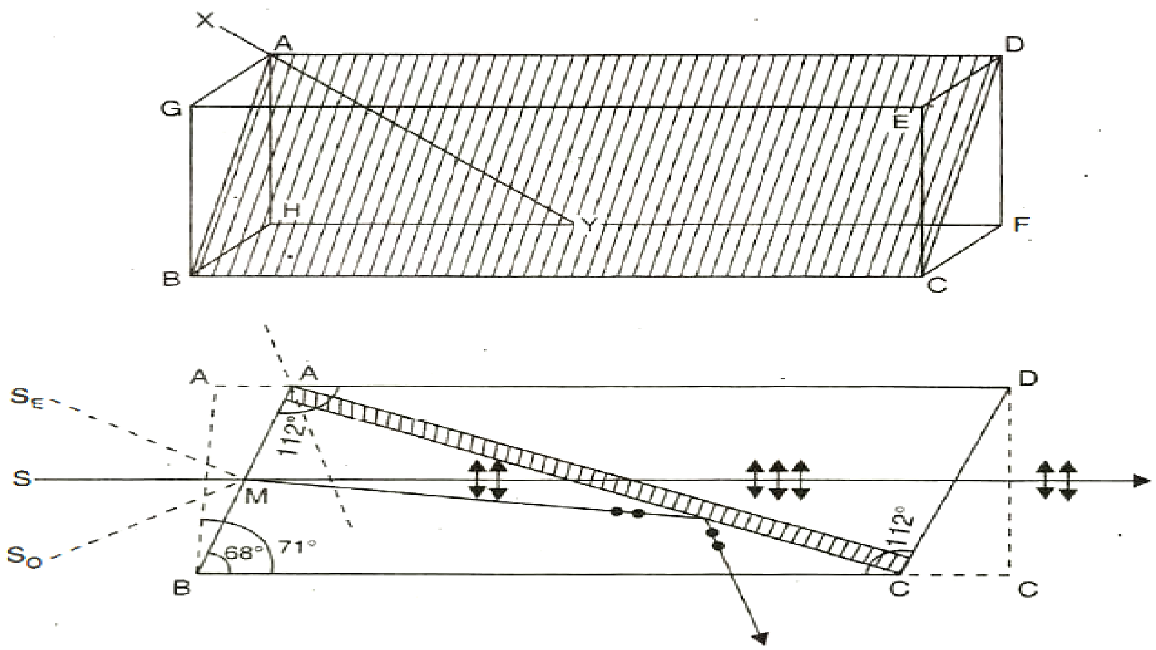
It is an optical device made from a calcite crystal for producing and analysing plane polarised light.

Principle:

It is based on the principle that it eliminates the ordinary ray by total internal so that the extraordinary ray became plane polarised emerges out from it. It is based on double refraction.

Construction:

A calcite crystal about the three times as long as the wide is taken .Its end faces are ground such that the angles in the principal section become 68° and 112^0 instead of 71° and 109°.The crystal is cut apart along a plane which is perpendicular to both the principal section .The two cut surfaces are ground of polished optically flat. They are then cemented together by Canada balsam whose refractive index is 1.55 for sodium light and the crystal is then enclosed in a tube blackened inside.



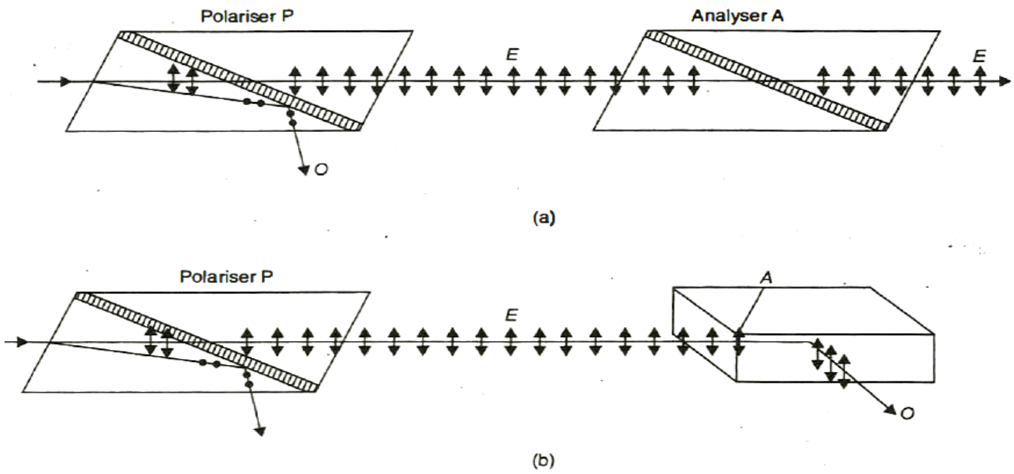
Action:

When a ray of unpolarised light is incident on the nicol prism it splits up into two refracted ray as O & E ray. Since the refractive index of the canabalsum 1.55 is less than the refractive index of calcite for the ordinary ray (O- ray), so the O- ray on reaching the Canada balsam get totally reflected and is absorbed by the tube containing the crystal while E-ray on reaching the Canada balsam is get transmitted. Since E- ray is plane polarised then the light emerging from the nicol is plane polarised in which vibration are parallel to the principal section.

Uses:

The nicol prism can be used both as apolariser and also an analyser.

When a ray of unpolarised light is incident on a nicol prism, then the ray emerging from the nicolprism is plane polarised with vibration in principal section. As this, ray falls on a second nicol which is parallel to that of 1st, its vibration will be in the principal section of 2nd and will be completely transmitted and the intensity of emergent light is maximum, thus the nicol prism behaves as a polariser.



If the second nicol is rotated such that its principal section is perpendicular to that of 1st then the vibration in the plane polarisation may incident on 2nd will be perpendicular to the principal section of 2nd.

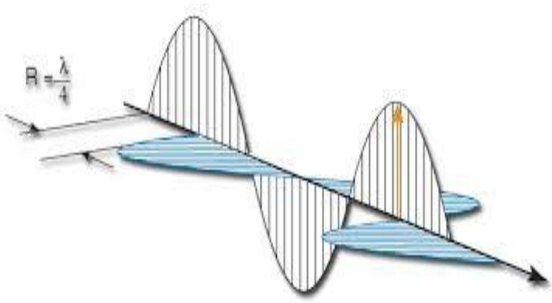
Hence the ray will behave as a ray inside the 2nd and will lost by total reflection at the balsam surface.

If the second nicol is further rotated to hold its principal section again parallel to that of 1st the intensity will be again maximum then the 1st prism acts as apolariser and the 2nd prism acts as an analyser.

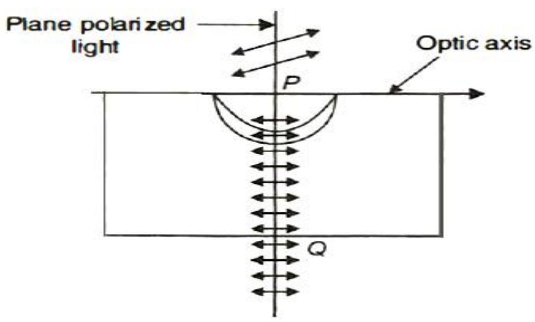
Limitations:

1. The nicol prism works only when the incident beam is slightly convergent or slightly divergent.
2. The angle of incidence must be confined with 14^0.

Quarter wave plate: A double refracting crystal plate having a thickness such as to produce a path difference of $\frac{\lambda}{4}$ or a phase difference of $\frac{\pi}{2}$ between the ordinary and extra ordinary wave is called as quarter wave plate or $\frac{\lambda}{4}$ plate .

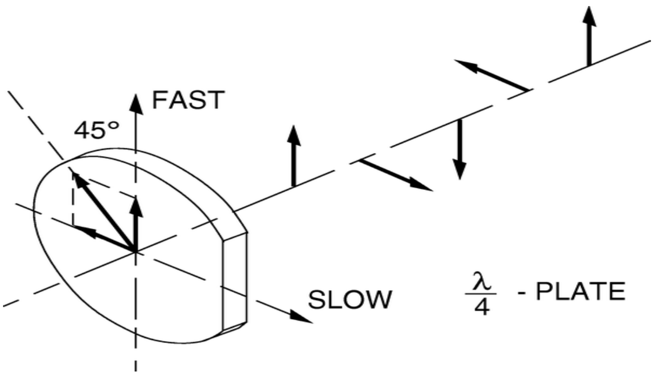


Construction: It can be constructed by cutting a plane from double refracting crystal such that its face parallel to the optic axis.



Working:

When a beam of monochromatic light incident on the plate it will be broken up into O-ray and E-ray which will perpendicular to the direction of wave propagation and vibrating in the direction of incidence respectively.



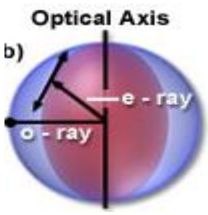
Let us consider a doubly refracting crystal

Let t= thickness of crystal plate
μ_O be the refractive index of the crystal for O-ray
μ_E be the refractive index of the crystal for O-ray

$\mu_O t$ = optical path for O ray
$\mu_E t$ = optical path for E ray
then the path difference between the waves is ($\mu_O - \mu_E$)t
if the plate acts as quarter wave plate ,
then ($\mu_O - \mu_E$)t = λ/4

$$\Rightarrow t = \frac{\lambda}{4(\mu_O - \mu_E)}$$

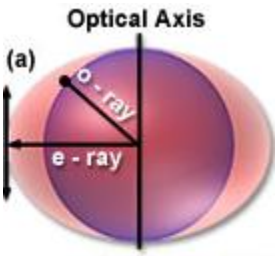
This is for positive crystal. The crystal in which the O-ray travels with a less velocity than E-ray called positive crystal.



For positive crystal $V_O \langle V_E$ and $\mu_E \rangle \mu_O$
Ex: calcite, tormulaline etc.

The crystal in which the O-ray travels with a greater velocity than E-ray called positive crystal.
For a-ve crystal and $V_O \rangle V_E$ and $\mu_E \langle \mu_O$

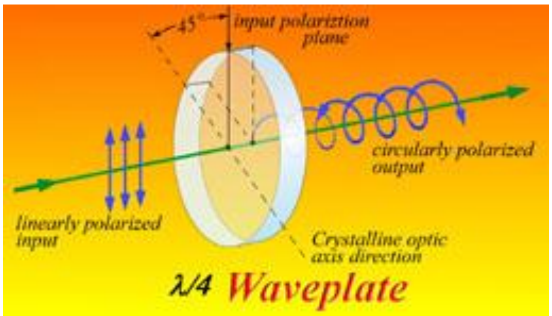
Ex: quartz, ice



$$t = \frac{\lambda}{4(\mu_E - \mu_O)}$$

Uses:

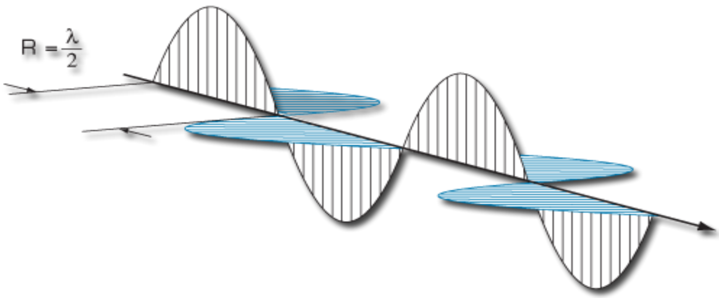
1. It is used for producing circularly and elliptically polarised light.



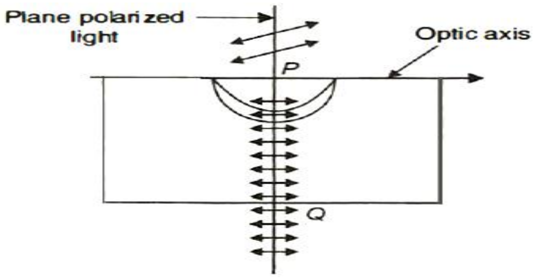
2. In addition with nicol prism it is used for analysing all kind of polarised light.

Half wave plate

A double refracting crystal plate having a thickness such as its produces a path difference of λ/2 between the ordinary and extraordinary wave is called half wave plate.

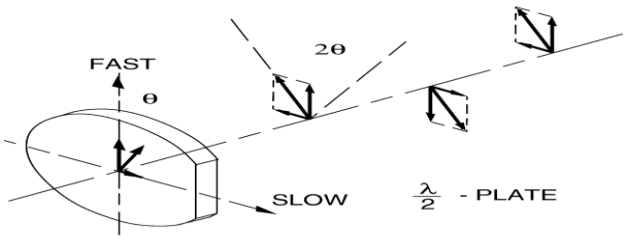


Construction: It can be constructed by cutting a plane from double refracting crystal such that its face parallel to the optic axis.



Working:

When a beam of monochromatic light incident on the plate it will be broken up into O-ray and E-ray which will perpendicular to the direction of wave propagation and vibrating in the direction of incidence respectively.

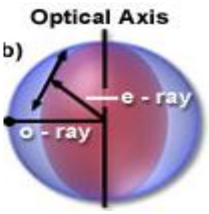


Let us consider a doubly refracting crystal

Let t= thickness of crystal plate
μ_O be the refractive index of the crystal for O-ray
μ_E be the refractive index of the crystal for O-ray
$\mu_O t$ = optical path for O ray
$\mu_E t$ = optical path for E ray
then the path difference between the waves is ($\mu_O - \mu_E$) t
If the plate acts as quarter plate, then ($\mu_O - \mu_E$) t = λ/2

$$\Rightarrow t = \frac{\lambda}{2(\mu_O - \mu_E)}$$

This is for positive crystal. The crystal in which the O-ray travels with a less velocity than E-ray called positive crystal.



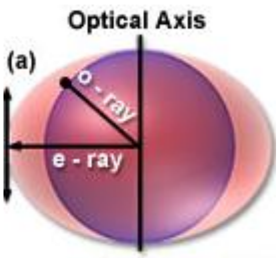
For positive crystal $V_O \langle V_E$ and $\mu_E \rangle \mu_O$

Ex: calcite, tormulaline etc.

The crystal in which the O-ray travels with a greater velocity than E-ray called positive crystal.

For a-ve crystal and $V_O \rangle V_E$ and $\mu_E \langle \mu_O$

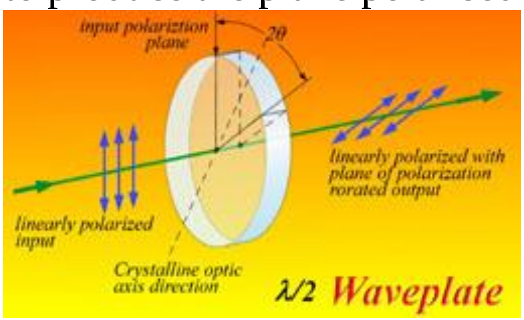
Ex: quartz, ice



$$t = \frac{\lambda}{2(\mu_E - \mu_O)}$$

Uses: 1.It is used in polarimeter as half shade devices to divide the field of view into two halves presented side by side

2.It is used to produce the plane polarised light.



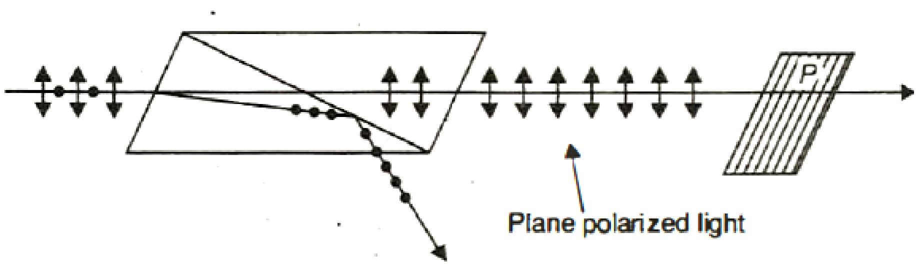
λ/4 plate	λ/2 plate

1.It produces a path difference of λ /4 between O and E wave 2. The light emerging from a λ /4 plate maybe circularly elliptically or plane polarised. 3. In this case nicol may give a non zero minimum. 4. It is used for production of all type polarised light.	1.It produces a path difference of λ /2 between O and E ray. 2. The light emerging from a λ /2 plates is plane polarised for all orientation of the plate. 3. In this case nicol may give a zero minimum always. 4. It is used in polarism for half shade device.

Production and Analysis Polarised Light

1. Production of plane polarised light:

To produce plane polarised light a beam of ordinary light is sent through a Nicol prism in a direction almost parallel to the long edge of the prism. Inside the prism the beam is broken upto two components 'O' and 'E' ray. The 'O' component is totally reflected at the Canada balsam and is absorbed.



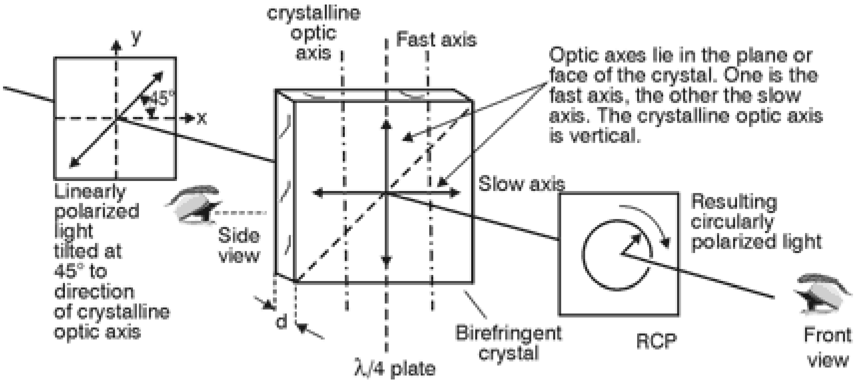
The 'E' component emerges out which is plane polarised with vibration parallel to the end faces of the Nicol.

2. Production of circularly polarised light:

The circularly polarised light can be produced by allowing plane-polarised light

obtained from the Nicol to fall normally on a quarter wave plate such that the

direction of vibration in the incident plane polarised light makes an angle of 45^0 with
the optic axis of the crystal.



Inside the plate the incident waves of amplitude A is divided into $E = A\cos 45^0$

$O = A\sin 45^0$ with a phase difference $\frac{\pi}{2}$ between them.

Let $A\cos 45^0 = A\sin 45^0$ Asin $45^0 = a$ of the axis of x

Let $x = a\sin(wt + \frac{\pi}{2}) = a\cos wt$ and $y = a\sin wt$

Eliminating t from both the equation, we have

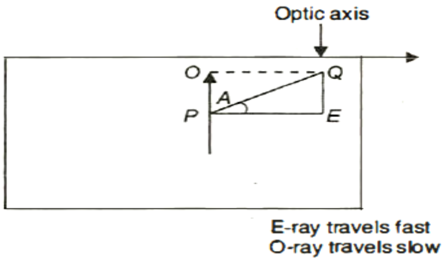
$x^2 + y^2 = a^2$ which represents a circle.

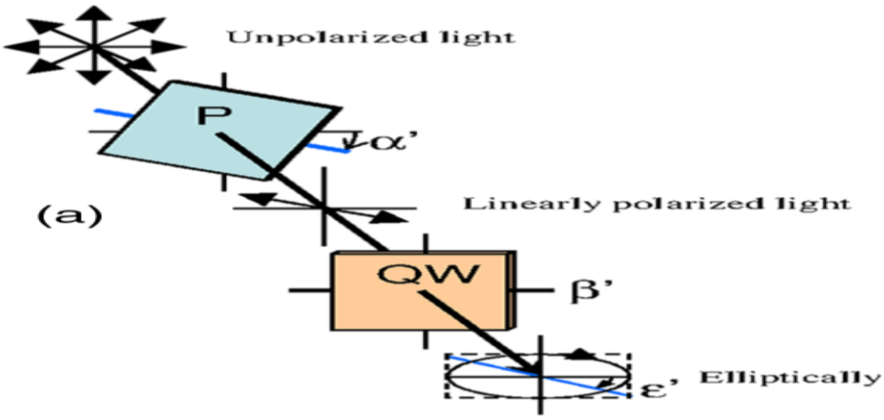
Hence the light emerging from $\lambda/4$ plate is circularly polarised.

3. Production of elliptically polarised light:

The elliptically polarised light can be produced by allowing plane polarised light normally in a quarter wave plate such that the direction of vibration in the incident plane polarised light makes an angle other than 0^0,45^0 and 90^0 with the optical axis which is 30^0.

In this case the incident wave is divided inside the plate into E and O components of unequal amplitude $A\cos 30^0$ and $A\sin 30^0$ respectively which emerge from the plate with a phase difference of $\frac{\pi}{2}$.





If we take $A\cos 30^0$ = a and $A\sin 30^0$ = b,then the emerging component can be written as,

$$x = a\sin(wt + \frac{\pi}{2}) = a\cos wt \text{ and } y = b\sin wt$$

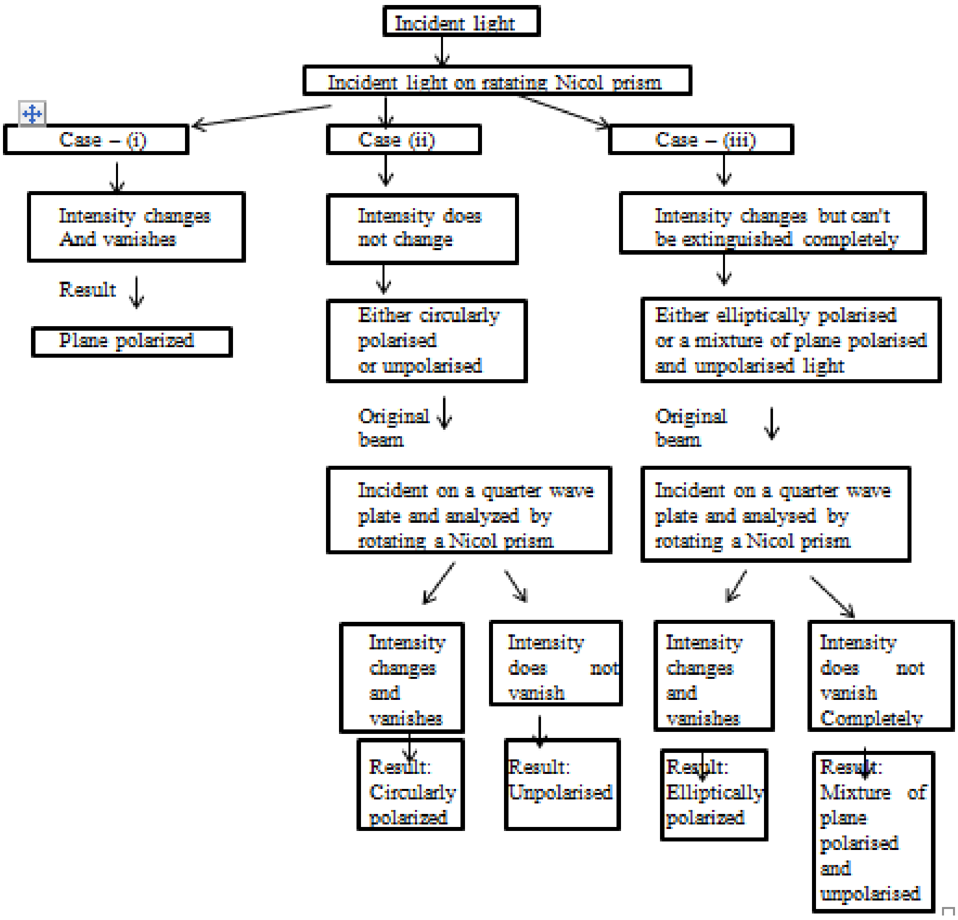
Now eliminating 't' from both the equation we have

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which the equation of an ellipse. Hence the emerging light coming from

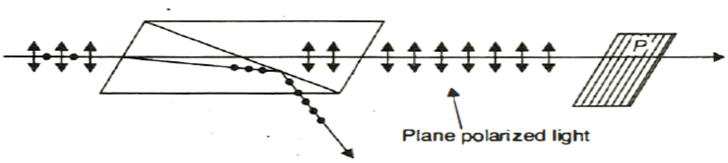
$\lambda/4$ plate is elliptically polarised

Analysis of different polarised light:

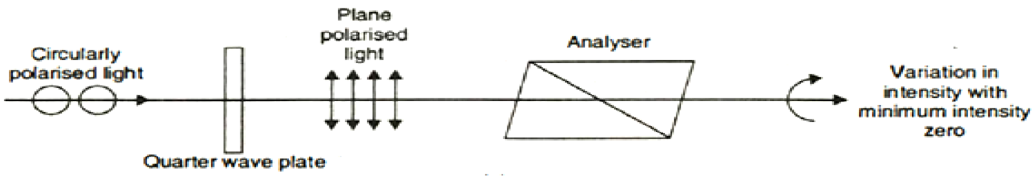
The whole analysis of different type of polarised light can be represented in algorithm form with figure as follows:



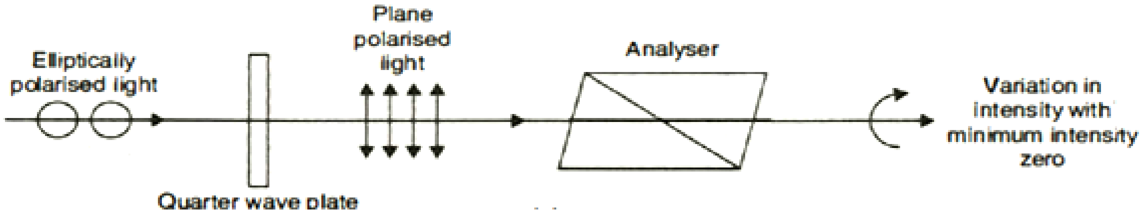
Case: 1



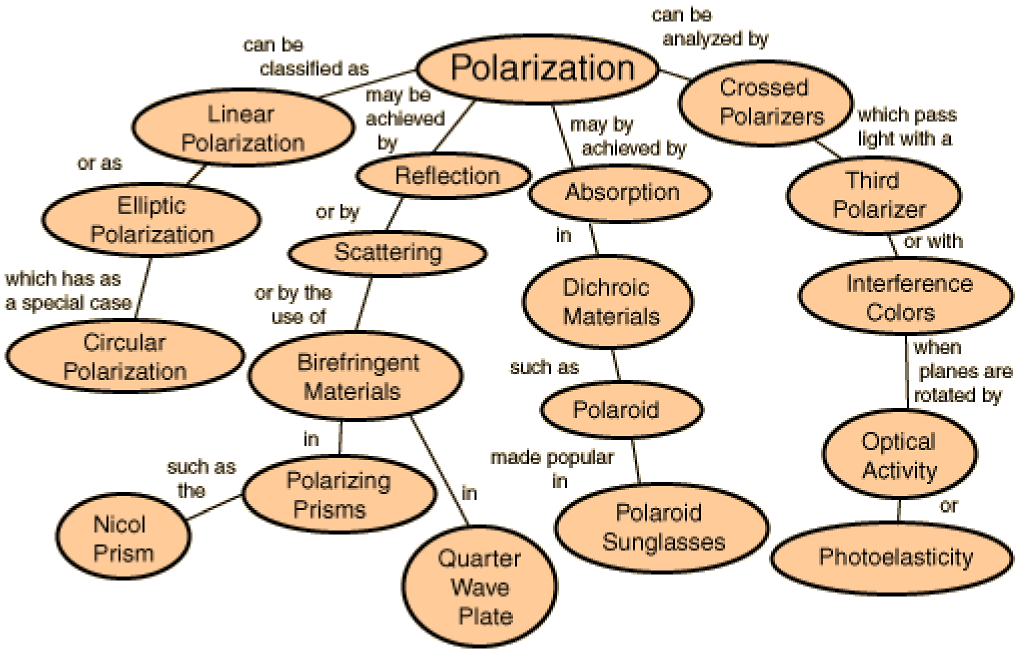
Case: 2



Case: 3



POLARISATION IN SUMMARY



VECTOR CALCULUS

The electric field $(\vec{E})$, magnetic induction $(\vec{B})$, magnetic intensity $(\vec{H})$, electric displacement $(\vec{D})$, electrical current density $(\vec{J})$, magnetic

vector potential $(\vec{A})$ etc. are, in general, functions of position and time. These are vector fields.

Scalar quantities such as electric potential, electric charge density, electromagnetic energy density etc. are also function of position and time. They are known of as fields.

Time Derivative of a Vector Field

If $\vec{A}(t) \rightarrow$ time dependent vector field, then the Cartesian coordinates

$$\vec{A}(t) = \hat{i}\vec{A}_x(t) + \hat{j}\vec{A}_y(t) + \hat{k}\vec{A}_z(t)$$

$$\frac{d\vec{A}}{dt} = \hat{i}\,\frac{\partial \vec{A}_x(t)}{\partial t} + \hat{j}\,\frac{\partial \vec{A}_y(t)}{\partial t} + \hat{k}\,\frac{\partial \vec{A}_z(t)}{\partial t}$$

Notes: $$\frac{d}{dt}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{dt} + (\frac{d\vec{A}}{dt})\times\vec{B}$$

Gradient of a Scalar Field

The change of a scalar field with position is described in terms of gradient operator.

$$grad(V) = \vec{\nabla}V = \hat{i}\,\frac{\partial V}{\partial x} + \hat{j}\,\frac{\partial V}{\partial y} + \hat{k}\,\frac{\partial V}{\partial z}$$

Where $\vec{\nabla} = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$ is **del operator or nabla**

$\vec{\nabla}V$ is a vector. The gradient of a scalar is a vector.

Divergence of a Vector Field

The divergence of a vector field $\vec{A}$ is given by

$$\vec{\nabla}.\vec{A} = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}).(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) = (\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \hat{k}\frac{\partial A_z}{\partial z})$$

Divergence of a vector field is a scalar.

Notes:

- $\vec{\nabla}.(\vec{A}+\vec{B}) = \vec{\nabla}.\vec{A} + \vec{\nabla}.\vec{B}$
- $\vec{\nabla}.(V\vec{A}) = (\vec{\nabla}V).\vec{A} + V(\vec{\nabla}.\vec{A})$ where V is a scalar field
- If the divergence of a vector field vanishes everywhere, it is called a solenoidal field.
- Divergence of a vector field is defined as the net outward flux of that field per unit volume at that point.

Curl of a Vector Field

The curl of a vector field is given by

$$\text{Curl A} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \end{vmatrix} = \hat{i}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{j}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$$

- Curl of a vector field is a vector
- If V is a scalar field, $\vec{A}$ and $\vec{B}$ are two vector fields, then

  $\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$

  $\vec{\nabla}\times(V\vec{A}) = (\vec{\nabla}V)\times\vec{A} + V(\vec{\nabla}\times\vec{A})$
- If curl of a vector field vanishes, then it is called an irrotational field.

Successive Operation of the $\vec{\nabla}$ operator

(i) <u>Laplacian</u>

$$\vec{\nabla}\times\vec{\nabla}=(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}).(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z})=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

$$\Rightarrow\vec{\nabla}^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

This is called Laplacian Operator

(ii) <u>Curl of gradient of ascalar</u>

$$\vec{\nabla}V=\hat{i}\frac{\partial V}{\partial x}+\hat{j}\frac{\partial V}{\partial y}+\hat{k}\frac{\partial V}{\partial z}$$

Where V is a scalar field

$$\vec{\nabla}\times(\vec{\nabla}V)=\hat{i}[(\frac{\partial}{\partial y})(\frac{\partial V}{\partial z})-(\frac{\partial}{\partial z})(\frac{\partial V}{\partial y})]+\hat{j}[(\frac{\partial}{\partial z})(\frac{\partial V}{\partial x})-(\frac{\partial}{\partial x})(\frac{\partial V}{\partial z})]+\hat{k}[(\frac{\partial}{\partial x})(\frac{\partial V}{\partial y})-(\frac{\partial}{\partial y})(\frac{\partial V}{\partial x})]$$

$$\because\vec{\nabla}\times(\vec{\nabla}V)=\begin{vmatrix}\hat{i} & \hat{j} & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x}\\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial x} & \frac{\partial V}{\partial x}\end{vmatrix}$$

$$\Rightarrow\vec{\nabla}\times(\vec{\nabla}V)=\hat{i}(\frac{\partial^2 V}{\partial y\partial z}-\frac{\partial^2 V}{\partial z\partial y})+\hat{j}(\frac{\partial^2 V}{\partial z\partial x}-\frac{\partial^2 V}{\partial x\partial z})+\hat{k}(\frac{\partial^2 V}{\partial x\partial y}-\frac{\partial^2 V}{\partial y\partial x})=0$$

Thus Curl of gradient of a scalar field is zero.

<u>Note:</u>

- If $\vec{\nabla}\times\vec{A}=0$, then $\vec{A}$ can be expressed as gradient of a scalar field i.e. $\vec{A}=\vec{\nabla}V$
- Conversely if a vector field is gradient of a scalar then its curl vanishes.

(iii) <u>Divergence of Curl of a Vector Field</u>

$$\vec{\nabla}\times\vec{A}=\hat{i}(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})+\hat{j}(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})+\hat{k}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})$$

$$\vec{\nabla}.\vec{\nabla}\times\vec{A}=(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}).[(\hat{i}(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})+\hat{j}(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})+\hat{k}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})]$$

$$\Rightarrow\vec{\nabla}.\vec{\nabla}\times\vec{A}=\frac{\partial}{\partial x}(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})+\frac{\partial}{\partial y}(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})+\frac{\partial}{\partial z}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})$$

$$\Rightarrow\vec{\nabla}.\vec{\nabla}\times\vec{A}=\frac{\partial^2 A_z}{\partial x\partial y}-\frac{\partial^2 A_y}{\partial x\partial z}+\frac{\partial^2 A_x}{\partial y\partial z}-\frac{\partial^2 A_z}{\partial y\partial x}+\frac{\partial^2 A_y}{\partial z\partial x}-\frac{\partial^2 A_x}{\partial z\partial y}$$

$$\Rightarrow \vec{\nabla}.\vec{\nabla}\times\vec{A} = 0$$

i.e. divergence of curl of a vector is zero.

Conversely, if the divergence of a vector field is zero, then the vector field can be expressed as the curl of a vector.

(iv) $\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2 A$

(v) $\vec{\nabla}.(\vec{A}\times\vec{B}) = \vec{B}.(\vec{\nabla}\times\vec{A}) - \vec{A}.(\vec{\nabla}\times\vec{B})$

Line Integral of a Vector

The line integral of a vector field between two points a and b, along a given path is

$$I_L = \int_a^b \vec{A}.\vec{dl}$$

$\vec{dl} \rightarrow$ elemental length along the given path between a and b.

The line integral of a vector field is a scalar quantity.

$$I_L = \int_a^b (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z).(\hat{i}dx + \hat{j}dy + \hat{k}dz) = \int_a^b (A_x dx + A_y dy + A_z dz)$$

Notes:

- If the integral is independent of path of integration between a and b, then the vector field is conservative field.
- The line integral of a conservative field $\vec{A}$ along a closed path vanishes

  i.e. $\oint \vec{A}.\vec{dl} = 0$
- In general, the line integral depends upon the path between a and b.

Surface integral of a Vector

The surface integral of a vector field $\vec{A}$, over a given surface $\vec{S}$ is

$$I_s = \int_S \vec{A}.\vec{ds}$$

Where $\vec{ds} \rightarrow$ elemental area of surface S

The direction of $\vec{ds}$ is along the outward normal to the surface.

Writting $\vec{ds} = \hat{n} ds$, where $\hat{n}$ is unit vector normal to the surface at a given point.

So $I_s = \int_S \vec{A}.\vec{ds} = \int_S \vec{A}.\hat{n} ds = \int_n A_n ds$

where $A_n \to \vec{A}.\hat{n}$, normal to the component of the vector at the area element.

So, surface integral of a vector field over a given area is equal to the integral of its normal component over the area.

Surface area of a vector field is a scalar.

Example: $\phi_E = \int_S \vec{E}.\vec{ds}$

Volume integral of a Vector

The volume integral of a vector field $\vec{A}$ over a given volume V is

$I_V = \int_V \vec{A} dV$

Where dV is the elemental volume (a scalar)

Volume integral of a vector field is a vector.

Gradient, Divergence and Curl in terms of Integrals

The gradient of a scalar field φ is the limiting value of its surface integral per unit volume, as volume tends to zero

i.e. $\nabla \phi = \lim_{\Delta V \to 0} \frac{\oiint \phi ds}{\Delta V}$

The divergence of a vector field A is the limiting value of its surface integral per unit volume, over an area enclosing the volume, as volume tends to zero.

$$\vec{\Delta}.\vec{A} = \lim_{\Delta V \to 0} \frac{\oiint_{\Delta S} \vec{A}.\vec{ds}}{\Delta V}$$

The curl of a vector field is the limiting value of its line integral along a closed path per unit area bounded by the path, as the area tends to zero,

$$\vec{\Delta} \times \vec{A} = \lim_{\Delta S \to 0} \frac{\oint \vec{A}.\vec{dl}}{\Delta S}$$

where $\hat{n}$ is the unit vector normal to the area enclosed.

Gauss Divergence Theorem

The volume integral of divergence of a vector $\vec{A}$ over a given volume V is equal to the surface integral of the vector over a closed area enclosing the volume.

$$\int_V \vec{\nabla}.\vec{A}dV = \oint_S \vec{A}.d\vec{S}$$

This theorem relates volume integral to surface integral.

Stokes Theorem

The surface integral of the curl of a vector field $\vec{A}$ over a given surface area S is equal to the line integral of the vector along the boundary C of the area

$$\int_S (\vec{\nabla}\times\vec{A}).d\vec{S} = \oint_C \vec{A}.d\vec{l}$$

For a closed surface C=0. Hence surface integral of the curl of a vector over a closed surface vanishes.

Green's Theorem

If there are two scalar functions of space f and g, then Green's theorem is used to change the volume integral into surface integral. This theorem is expressed as

$$\int_V (f\nabla^2 g - g\nabla^2 f)dV = \int_S (f\nabla g - g\nabla f)dS$$

V- volume enclosed by surface S.

Electric Polarization ($\vec{P}$)

Electric polarization $\vec{P}$ is defined as the net dipole moment ($\vec{p}$) induced in a specimen per unit volume.

$$\vec{P} = \frac{\vec{p}}{V}$$

Unit is 1 coul/m^2

The dipole moment is proportional to the applied electric field.

So $\vec{p} = \alpha\vec{E}$, $\quad \alpha \to proportionality\ cons\tan t, known\ as\ polarizability$

If N is the number of molecules per unit volume then polarization is given by

$$\vec{P} = N\alpha\vec{E}$$

Electric Displacement Vector $\vec{D}$

The electric displacement vector $\vec{D}$ is given by

$$\vec{D} = \vec{P} + \varepsilon_0\vec{E} \text{ ------------------- (1)}$$

where is the $\vec{P} \to \textit{polarization vector}$

Unit of $\vec{D} \to 1\dfrac{ampere \times \sec}{m^2}$

In linear and isotropic dielectric,

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_0\varepsilon_r\vec{E} \text{ ---------------(2)}$$

Comparing equations (1) and (2), we get

$$\varepsilon_0\varepsilon_r\vec{E} = \vec{P} + \varepsilon_0\vec{E}$$

$$\Rightarrow \vec{P} = \varepsilon_0(\varepsilon_r - 1)\vec{E}$$

Electric Flux (φ_E)

The number of lines of force passing through a given area is known as electric flux.
It is given by

$$\phi_E = \int_S \vec{E}.\vec{dS}$$

Unit of flux-$1\dfrac{N \times m^2}{Coul}$

Gauss' Law in Electrostatic:

Statement: The total electric flux (φ_E) over a closed surface is equal to $\dfrac{1}{\varepsilon_0}$ times the net charge enclosed by the surface.

$$\phi_E = \int_S \vec{E}.\vec{dS} = \frac{q_{net}}{\varepsilon_0}$$

Here S is known as Gaussian surface.

In a dielectric medium Gauss' law is given by

$$\phi_E = \int_S \vec{E}.\vec{dS} = \frac{q_{net}}{\varepsilon}$$

ε - Permittivity of the medium.

In terms of displacement vector Gauss' law is given by

$$\phi_E = \int_S \vec{D}.\vec{dS} = q_{net}$$

Notes:

- The charges enclosed by the surface may be point charges or continuous charge distribution.
- The net electric flux may be outward or inward depending upon the sign of charges.
- Electric flux is independent of shape & size of Gaussian surface.
- The Gaussian surface can be chosen to have a suitable geometrical shape for evaluation of flux.
- Limitation of Gauss' Law
  (a) Since flux is a scalar quantity Gauss' law enables us to find the magnitude of electric field only.
  (b) The applicability of the law is limited to situations with simple geometrical symmetry.

Gauss' Law in Differential form

Gauss' law is given by

$$\int_S \vec{E}.\vec{dS} = \frac{q_{net}}{\varepsilon}$$

For a charge distribution

$$q_{net} = \int_V \rho \, dV \qquad where\ \rho \rightarrow volume\ charge\ density$$

Using Gauss divergence theorem

$$\oint_S \vec{E}.\vec{dS} = \int_V \vec{\nabla}.\vec{E}\, dV$$

So $$\frac{1}{\varepsilon_0}\int_V \rho \, dV = \int_V \vec{\nabla}.\vec{E}\, dV$$

Or $$\int_V (\vec{\nabla}.\vec{E}-\frac{\rho}{\varepsilon_0})\,dV=0$$

$$\Rightarrow \quad \vec{\nabla}.\vec{E}-\frac{\rho}{\varepsilon_0}=0$$

$$\Rightarrow \quad \vec{\nabla}.\vec{E}=\frac{\rho}{\varepsilon_0}$$

This is the differential form of Gauss' law.

Magnetic Intensity (H) and Magnetic Induction $(\vec{B})$

The magnetic intensity $(\vec{H})$ is related to the magnetic field induction $(\vec{B})$ by

$$(\vec{H})=\frac{(\vec{B})}{\mu_0}$$

Unit: in SI system $(\vec{H})$ is in amp/m and $(\vec{B})$ in tesla.

Magnetic Flux (ϕ_m)

The magnetic flux over a given surface area S is given by

$$(\phi_m)=\int_S \vec{B}.\vec{dS}=\int_S B\,dS\cos\alpha$$

where $\alpha \rightarrow$ *angle between magnetic field* $\vec{B}$ *and normal to the surface*

Unit of flux: 1 weber in SI

1 maxwell in cgs(emu)

So 1T= 1 weber/m^2

1 gauss= 1maxwell/cm^2

Gauss' Law in magnetism

Since isolated magnetic pole does not exist, by analogy with Gauss' law of electrostatics, Gauss' law of magnetism is given by

$$\oint_S \vec{B}.\vec{dS}=0$$

Using Gauss divergence theorem

$$\oint_S \vec{B}.\vec{dS}=\int_V \vec{\nabla}.\vec{B}\,dV=0$$

$$\Rightarrow \quad \vec{\nabla}.\vec{B}=0$$

This is the differential form of Gauss' law of magnetism.

Ampere's Circuital law

Statement:-The line integral of magnetic field along a closed loop is equal to $\mu_0$timesthenetelectriccurrentenclosedbyloop.

$$\oint_C \vec{B}.\vec{dl} = \mu_0 I$$

Where $I \rightarrow$ *net current enclosed by the loop*

$C \rightarrow$*closed path enclosing the current (called ampere loop).*

In terms of magnetic intensity

$$\oint_C \vec{H}.\vec{dl} = I$$

Ampere's Law in Differential form

Ampere's law is

$$\oint_C \vec{B}.\vec{dl} = \mu_0 I \text{ ---------------------------------(i)}$$

Using Stoke's theorem, we have

$$\oint_C \vec{B}.\vec{dl} = \int_S (\vec{\nabla} \times \vec{B}).\vec{ds} \text{ -------------------------(ii)}$$

In terms of current density J

$$\mu_0 I = \mu_0 \int_S \vec{J}.\vec{ds} \text{ -------------------------------------(iii)}$$

Using (ii) and (iii) in equation (i) we have

$$\int_S (\vec{\nabla} \times \vec{B}).\vec{ds} = \mu_o \int_S \vec{J}.\vec{ds} = \int_S (\mu_o \vec{J}).\vec{ds}$$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \mu_o \vec{J}$$

This is Ampere's circuital law in differential form.

Faraday's Law of electromagnetic induction

Statement :-The emf induced in a conducting loop is equal to the negative of rate of change of magnetic flux through the surface enclosed by the loop.

$$\varepsilon = -\frac{\partial \phi_m}{\partial t}$$ --------------(i)

The induced emf is the line integral of electric field along the loop.

$$\varepsilon = \oint_C \vec{E}.\vec{dl}$$

The magnetic flux is

$$\phi_m = \int_S \vec{B}.\vec{ds}$$

So from the above

$$\oint_C \vec{E}.\vec{dl} = -\frac{\partial}{\partial t}\int_S \vec{B}.\vec{ds}$$

This is Faraday's law of electromagnetic induction in terms of $\vec{E}$ and $\vec{B}$

Differential form of Faraday's Law

Now using Stokes' theorem

$$\oint_C \vec{E}.\vec{dl} = \int_S (\vec{\nabla}\times\vec{E}).\vec{ds}$$

But $\oint_C \vec{E}.\vec{dl} = -\frac{\partial \phi_m}{\partial t} = -\frac{\partial}{\partial t}\int_S \vec{B}.\vec{ds}$

From above two equations

$$\int_S (\vec{\nabla}\times\vec{E}).\vec{ds} = -\frac{\partial}{\partial t}\int_S \vec{B}.\vec{ds} = -\int_S \frac{\partial \vec{B}}{\partial t}.\vec{ds}$$

Or $\int_S (\vec{\nabla}\times\vec{E}+\frac{\partial \vec{B}}{\partial t}).\vec{ds}=0$

$$\Rightarrow \quad \vec{\nabla}\times\vec{E}+\frac{\partial \vec{B}}{\partial t}=0$$

This is differential form of Faraday's law electromagnetic induction.

Equation of Continuity

The electric current through a closed surface S is

$$I=\oint_S \vec{J}.\vec{ds}$$

Using Gauss divergence theorem

$$I=\oint_S \vec{J}.\vec{ds}=\int_V \vec{\nabla}.\vec{J}\ dV \text{ ---------------- (i)}$$

Where S is boundary of volume V.

Now $I=-\frac{\partial q}{\partial t} \quad \rightarrow \quad \textit{rate of decrease of charge from the volume through surface S}$

$$\Rightarrow \quad I=-\frac{\partial}{\partial t}\int_V \rho\, dV=-\int_V \frac{\partial \rho}{\partial t}dV \quad \text{----------------------(ii)}$$

From (i) and (ii)

$$\int_V \vec{\nabla}.\vec{J}\ dV=-\int_V \frac{\partial \rho}{\partial t}dV$$

$$\Rightarrow \quad \int_V (\vec{\nabla}.\vec{J}+\frac{\partial \rho}{\partial t})dV=0$$

$$\Rightarrow \quad \vec{\nabla}.\vec{J}+\frac{\partial \rho}{\partial t}=0$$

This is equation of continuity.

Displacement Current

Maxwell associated a current (known as displacement current) with the time varying electric field.

A parallel plate capacitor connected to a cell is considered. During charging field $\vec{E}$ between varies.

Let $q \rightarrow$ instantaneous charge on capacitor plates.

$A \rightarrow$ area of each plate

We know that the electric field between the capacitor plates is

$$E = \frac{q}{\varepsilon_0 A}$$

$$\Rightarrow \quad \frac{dE}{dt} = \frac{1}{\varepsilon_0 A}\frac{dq}{dt}$$

$$\Rightarrow \quad \varepsilon_0 A\frac{dE}{dt} = \frac{dq}{dt}$$

$\Rightarrow \quad I_d = \varepsilon_0 A\dfrac{dE}{dt} \quad where\ I_d \rightarrow$ displacement current between the plates

I_d exists till $\vec{E}$ varies with time.

In general, whenever there is a time-varying electric field, a displacement current exists,

$$I_d = \varepsilon_0 \frac{\partial}{\partial t}\int_S \vec{E}.\vec{ds} = \varepsilon_0 \frac{\partial \phi_E}{\partial t}$$

Where ϕ_E is electric flux.

Modification of Ampere's circuital law

Taking displacement current into account Ampere's Circuital law is modified as

$$\oint_C \vec{B}.\vec{dl} = \mu_o \left(I + I_d \right)$$

This law is sometimes referred as Ampere- Maxwell law.

The corresponding differential form is given as,

$$\vec{\nabla}\times\vec{B}=\mu_o\left[\vec{J}+\varepsilon_o\frac{\partial\vec{E}}{\partial t}\right]$$

Or

$$\vec{\nabla}\times\vec{H}=\left[\vec{J}+\frac{\partial\vec{D}}{\partial t}\right]$$

By using

$$\varepsilon_0\vec{E}=D,\frac{B}{\mu_0}=H$$

Here $\varepsilon_0\frac{\partial\vec{E}}{\partial t}=J_d\rightarrow displacement\,current\,density$

<u>Distinction between displacement current and conduction current</u>

<u>Conduction current</u>	<u>Displacement current</u>
(i) Due to actual flow of charge in conducting medium. (ii) It obeys ohm's law. (iii) Depends upon V and R	(i) Exists in vacuum or any medium even in absence of free charge carriers. (ii) Does not obey ohm's law. (iii) Depend upon ϵ and $\frac{\partial E}{\partial t}$

<u>Relative magnitudes of displacement current and conduction current</u>

Let $E=E_0 Sin\omega t\rightarrow alternating\ field$

Then current density

$$J=\sigma E=\sigma E_0 Sin\omega t----------(i)$$

Displacement current density

$$J_d=\varepsilon_0\frac{\partial E}{\partial t}=\varepsilon_0\frac{\partial}{\partial t}\left(E_0 Sin\omega t\right)=\omega\varepsilon_0 E_0 Cos\omega t---------(ii)$$

Thus there is a phase difference of $\frac{\Pi}{2}$ between current density and displacement current density.

The ratio of their peak values

$$\frac{(J)_{\max}}{(J_d)_{\max}} = \frac{\sigma E_0}{\omega \varepsilon_0 E_0} = \frac{\sigma}{\omega \varepsilon_0}$$

It means this ratio depends upon frequency of alternating field.

Notes:

- For copper conductor the ratio is $\approx \frac{10^{19}}{\omega}$
- For f>10^{20} Hz, displacement current is dominant. So normal conductors behave as dielectric at extremely high frequencies.

Maxwell's Equations

The Maxwell's electromagnetic equations are

$$\vec{\nabla}.\vec{D} = \rho \qquad \text{--------------(1)}$$

$$\vec{\nabla}.\vec{B} = 0 \qquad \text{-----------------(2)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{------------------(3)}$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \qquad \text{-------------------(4)}$$

Notes:

- ➢ Equation (1) is the differential form of Gauss' law of electrostatics.
- ➢ Equation (2) is the differential form of Gauss' law of magnetism.
- ➢ Equation (3) is the differential form of Faraday's law of electromagnetic induction.

- Equation (4) is the generalized form of Ampere's circuital law.
- Equations (2) and (3) have the same form in vacuum and medium. They are also unaffected by the presence of free charges or currents. They are usually called the constraint equation for electric and magnetic fields.
- Equations (1) and (4) depend upon the presence of free charges and currents and also the medium.
- Equations (1) and (2) are called steady state equations as they do not involve time dependent fields.

<u>Maxwell's Equations in terms of E and B</u>

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{--------------(1)}$$

$$\vec{\nabla}.\vec{B} = 0 \quad \text{-----------------(2)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{------------------(3)}$$

$$\vec{\nabla}\times\vec{B} - \mu\varepsilon\frac{\partial \vec{E}}{\partial t} = \mu\vec{J} \quad \text{-------------------(4)}$$

In absence of charges

$$\vec{\nabla}.\vec{E} = 0 \quad \text{--------------(1)}$$

$$\vec{\nabla}.\vec{B} = 0 \quad \text{-----------------(2)}$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \text{------------------(3)}$$

$$\vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} = 0 \quad \text{-------------------(4)}$$

<u>Maxwell's Equations in Integral Form</u>

$$\oint_S \vec{E}.\vec{dS} = \frac{1}{\varepsilon_0}\int_V \rho\, dV \quad \text{--------------(1)}$$

$$\oint_S \vec{B}.\vec{dS} = 0 \qquad \text{--------------(2)}$$

$$\oint_C \vec{E}.\vec{dl} = -\frac{\partial}{\partial t}\oint_S \vec{B}.\vec{dS} \qquad \text{--------------(3)}$$

$$\oint_C \vec{B}.\vec{dl} = \mu_0 \int_S (\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}).\vec{dS} \qquad \text{--------------(4)}$$

Physical Significance of Maxwell's Equation

(i) Maxwell equations incorporate all the laws of electromagnetism.

(ii) Maxwell equations lead to the existence of electromagnetic waves.

(iii) Maxwell equations are consistent with the special theory of relativity.

(iv) Maxwell equations are used to describe the classical electromagnetic field as well as the quantum theory of interaction of charged particles electromagnetic field.

(v) Maxwell equations provided a unified description of the electric and magnetic phenomena which were treated independently.

Electromagnetic Waves

Wave Equation of electromagnetic wave in free space

In vacuum, in absence of charges, Maxwell's equations are

$$\vec{\nabla}.\vec{E} = 0 \qquad \text{--------------(1)}$$

$$\vec{\nabla}.\vec{B} = 0 \qquad \text{-----------------(2)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{-------------------(3)}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} \qquad \text{--------------------(4)}$$

Taking curl of equation (3)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

Using equation (4)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}) = -\varepsilon_0\mu_0\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \quad \vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2\vec{E} = -\varepsilon_0\mu_0\frac{\partial^2 \vec{E}}{\partial t^2}$$

Since $\vec{\nabla}(\vec{\nabla}.\vec{E}) = 0, \quad \nabla^2\vec{E} = \varepsilon_0\mu_0\frac{\partial^2 \vec{E}}{\partial t^2}$

Taking $\varepsilon_0\mu_0 = \frac{1}{c^2}, where \quad c \to velocity\ of\ light$

We have $\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$

This is the wave equation for $\vec{E}$.

Now taking curl of equation (4)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

Using equation (3)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\frac{\partial}{\partial t}(-\frac{\partial \vec{B}}{\partial t}) = -\varepsilon_0\mu_0\frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\Rightarrow \quad \vec{\nabla}(\vec{\nabla}.\vec{B}) - \nabla^2\vec{B} = -\varepsilon_0\mu_0\frac{\partial^2 \vec{B}}{\partial t^2}$$

Since $\vec{\nabla}(\vec{\nabla}.\vec{B}) = 0, \quad \nabla^2 \vec{B} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2}$

Taking $\varepsilon_0 \mu_0 = \frac{1}{c^2}, where \quad c \rightarrow velocity\ of\ light$

We have $\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$

This is the wave equation for $\vec{B}$.

The general wave equation in vacuum can be written as

$$\nabla^2 \vec{\Psi} = \frac{1}{c^2} \frac{\partial^2 \vec{\Psi}}{\partial t^2}$$

Where $\vec{\Psi} = \vec{E}\ or\ \vec{B}$

For charge free non-conducting medium, the general equation will be

$$\nabla^2 \vec{\Psi} = \frac{1}{v^2} \frac{\partial^2 \vec{\Psi}}{\partial t^2}$$

$\varepsilon\mu = \frac{1}{v^2}, where \quad v \rightarrow velocity\ of\ light\ in\ medium$s

Magnetic Vector Potential

The vector potential in a vector field is defined as when the divergence of a vector field is zero the vector can be expressed as the curl of a potential called vector potential ($\vec{A}$).

We know that $\vec{\nabla}.\vec{B} = 0$ (Maxwell equation)

Then $\vec{B} = \vec{\nabla} \times \vec{A} \quad (as\ div. of curl\ of\ a\ vector\ is\ zero)$

The vector $\vec{A}$ is called magnetic vector potential. The vector $\vec{A}$ can be chosen arbitrarily as addition of a constant vector or gradient of a scalar do not change the result.

Scalar Potential

The scalar potential in a scalar field is defined as when the curl of a field is zero the vector can be expressed as the negative gradient of a potential called scalar potential (ϕ).

We have $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Maxwell's equation 3)

Putting $\vec{B} = \vec{\nabla}\times\vec{A}$ in above we get

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$\Rightarrow \vec{\nabla}\times\left[\vec{E}+\frac{\partial\vec{A}}{\partial t}\right]=0$$

We know that curl of grad of a scalar is zero. So we can write

$$\vec{E}+\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi \text{ where } \phi \text{ is a scalar function called the scalar potential.}$$

So $\vec{E} = -\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\phi$

For a time independent field $\frac{\partial\vec{A}}{\partial t}=0$; *so*

$$\vec{E} = -\vec{\nabla}\phi \quad \text{here } \phi \rightarrow \text{electrostatic potential}$$

Wave equation in terms of scalar & vector potential

Let us consider the Maxwell's equations,

$$\left.\begin{aligned}&\vec{\nabla}.\vec{E}=0 \quad --------------(1)\\ &\vec{\nabla}\times\vec{B} = -\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad --------------(2)\end{aligned}\right\}$$

In free space and absence of charge

Writing $E = -\nabla\phi-\frac{\partial\vec{A}}{\partial t}$; $\vec{A}\rightarrow$ *vector potential*

We have

$$\vec{\nabla}.\vec{E} = \vec{\nabla}.\left(-\nabla\phi - \frac{\partial A}{\partial t}\right) = 0$$

$$or\ \nabla^2\phi + \frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{A}\right) = 0$$

Using Lorentz gauge condition

$$\vec{\nabla}.\vec{A} = -\frac{1}{c^2}\frac{\partial^2\phi}{\partial^2 t} = 0$$

$$We\,have \quad \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial^2 t} = 0$$

This is the wave equation in terms of scalar potential.

$$Putting \quad \vec{E} = -\nabla\phi - \frac{\partial A}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad in\ equation\ (2)\ we\ get$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \varepsilon_0\mu_0\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial\vec{A}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2\vec{A} = -\varepsilon_0\mu_0\frac{\partial}{\partial t}\left(\nabla\phi\right) - \varepsilon_0\mu_0\frac{\partial^2 A}{\partial t^2}$$

$$\Rightarrow \vec{\nabla}\left(\vec{\nabla}.\vec{A} + \varepsilon_0\mu_0\frac{\partial\phi}{\partial t}\right) = \nabla^2\vec{A} - \varepsilon_0\mu_0\frac{\partial^2 A}{\partial t^2}$$

The LHS vanishes by Lorentz gauge condition.

$$So \quad \nabla^2\vec{A} - \varepsilon_0\mu_0\frac{\partial^2 A}{\partial t^2} = 0$$

This is the wave equation in terms of vector potential.

Lorentz gauge potential

$$\nabla^2.\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \quad (Lorentz\ gauge\ condition)$$

$$\vec{\nabla}.\vec{A} = 0 \quad (Coulomb\ gauge\ condition)$$

Transverse nature of elecromagnetic wave

$The\ plane\ wave\ solution\ of\ wave\ equation\ for\ \vec{E}\ and\ \vec{B}\ are$

$$\vec{E}\left(\vec{r},t\right)=\hat{e}E_0e^{i(\vec{k}.\vec{r}-\omega t)}\quad --------(1)$$

$$\vec{B}\left(\vec{r},t\right)=\hat{b}B_0e^{i(\vec{k}.\vec{r}-\omega t)}\quad --------(2)$$

$where\quad \hat{e},\hat{b}\rightarrow unit\ vector\ along\ \vec{E}\ and\ \vec{B}\ respectively.$

$E_0, B_0 \rightarrow amplitudes\ of\ \vec{E}\ and\ \vec{B}\ respectively.$

$\vec{k}\rightarrow wave\ propagation\ vector$

$\omega\rightarrow angular\ \ frequency$

Using $\vec{\nabla}.\vec{E}=0$ in equation (1) we have

$$\vec{\nabla}.\left[\hat{e}E_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]=0$$

$$\Rightarrow \hat{e}.\vec{\nabla}\left[E_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]+\vec{\nabla}.\hat{e}\left[E_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]=0$$

$$\left\{as\ \vec{\nabla}.\left(V\vec{A}\right)=\left(\vec{\nabla}V\right).\vec{A}+V\vec{\nabla}.\vec{A}\right\}$$

Since $\hat{e}\rightarrow$ constant, $\vec{\nabla}.\hat{e}=0$

$$\hat{e}.\vec{\nabla}\left[E_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]=0$$

$$or\quad \hat{e}.i\vec{k}E_0e^{i(\vec{k}.\vec{r}-\omega t)}=0$$

$$Since\ E_0\neq 0,\quad e^{i(\vec{k}.\vec{r}-\omega t)}\neq 0,$$

$$\hat{e}.\vec{k}=0\quad ----------(3)$$

This shows the transverse nature of electric field.

Similarly, from Maxwell’s equation

$$\vec{\nabla}.\vec{B}=0$$

We have

$$\vec{\nabla}.\left[\hat{b}B_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]=0$$

$$\Rightarrow \hat{b}.\vec{\nabla}\left[B_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]+\vec{\nabla}.\hat{b}\left[B_0e^{i(\vec{k}.\vec{r}-\omega t)}\right]=0$$

Since $\hat{b}\rightarrow$ constant, $\vec{\nabla}.\hat{b}=0$

$$\hat{b}.\vec{\nabla}\left[B_0 e^{i(\vec{k}.\vec{r}-\omega t)}\right]=0$$

$$or \quad \hat{b}.i\vec{k}B_0 e^{i(\vec{k}.\vec{r}-\omega t)}=0$$

$$Since\ B_0 \neq 0, \quad e^{i(\vec{k}.\vec{r}-\omega t)} \neq 0,$$

$$\hat{b}.\vec{k}=0 \quad -----------(4)$$

This shows the transverse nature of magnetic field.

<u>Mutual orthogonality of **E**, **B** and **k**</u>

Now from Maxwell's 3rd equation we have

$$\vec{\nabla}\times[\hat{e}E_0 e^{i(\vec{k}.\vec{r}-\omega t)}]=-\frac{\partial}{\partial t}[\hat{b}B_0 e^{i(\vec{k}.\vec{r}-\omega t)}] \text{-----------}(5)$$

Using $\vec{\nabla}\times(\vec{A}V)=V(\vec{\nabla}\times\vec{A})+(\vec{\nabla}V)\times\vec{A}$, we have

$$\vec{\nabla}\times[\hat{e}E_0 e^{i(\vec{k}.\vec{r}-\omega t)}]=E_0 e^{i(\vec{k}.\vec{r}-\omega t)}(\vec{\nabla}\times\hat{e})+[\vec{\nabla}(E_0 e^{i(\vec{k}.\vec{r}-\omega t)})]\times\hat{e}$$

Since $\hat{e}$ is a constant unit vector, $(\vec{\nabla}\times\hat{e})=0$ and

$\vec{\nabla}(E_0 e^{i(\vec{k}.\vec{r}-\omega t)})=E_0 i\vec{k}e^{i(\vec{k}.\vec{r}-\omega t)}$ we get

$$\vec{\nabla}\times[\hat{e}E_0 e^{i(\vec{k}.\vec{r}-\omega t)}]=E_0 i\vec{k}e^{i(\vec{k}.\vec{r}-\omega t)}\times\hat{e}=E_0 ie^{i(\vec{k}.\vec{r}-\omega t)}(\vec{k}\times\hat{e})$$

Now $\frac{\partial}{\partial t}[\hat{b}B_0 e^{i(\vec{k}.\vec{r}-\omega t)}]=\hat{b}B_0\frac{\partial}{\partial t}\{e^{i(\vec{k}.\vec{r}-\omega t)}\}=\hat{b}B_0 e^{i(\vec{k}.\vec{r}-\omega t)}(-i\omega)$

Then from eqn. 5

$$E_0 ie^{i(\vec{k}.\vec{r}-\omega t)}(\vec{k}\times\hat{e})=-\hat{b}B_0 e^{i(\vec{k}.\vec{r}-\omega t)}(-i\omega)=\hat{b}B_0 i\omega e^{i(\vec{k}.\vec{r}-\omega t)}$$

$$\Rightarrow E_0(\vec{k}\times\hat{e})=\hat{b}B_0\omega$$

$$\Rightarrow (\vec{k}\times\hat{e})=\frac{B_0\omega}{E_0}\hat{b}$$

So $\hat{b}$ is perpendicular to both $\vec{k}$ *and* $\hat{e}$.

Thus electric field, magnetic field and propagation vector are mutually orthogonal.

Relative magnitudes of $\vec{E}$ *and* $\vec{B}$

Now taking magnitudes

$$\left|(\vec{k}\times\hat{e})\right| = \left|\frac{B_0\omega}{E_0}\hat{b}\right|$$

$$\Rightarrow k = \frac{\omega B_0}{E_0}$$

$$\Rightarrow \frac{E_0}{B_0} = \frac{\omega}{k} = c, \quad where\, c \rightarrow velocity\, of\ light$$

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Now using $B_0 = \mu_0 H_0$

$$\frac{E_0}{H_0} = \mu_0 c = \mu_0 \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{\mu_0}{\varepsilon_0} = Z_0$$

The quantity Z_0 has the dimension of electrical resistance and it is called the impedance of vacuum.

Phase relation between $\vec{E}$ *and* $\vec{B}$

In an electromagnetic wave electric and magnetic field are in phase.

Either electric field or magnetic field can be used to describe the electromagnetic wave.

Electromagnetic Energy Density

The electric energy per unit volume is

$$u_E = \frac{1}{2}\vec{E}.\vec{D} = \frac{1}{2}\varepsilon E^2 \quad --------(1)$$

The magnetic energy per unit volume is

$$u_B = \frac{1}{2}\vec{B}.\vec{H} = \frac{1}{2}\mu H^2 \quad --------(2)$$

The electromagnetic energy density is given by

$$u_{EM} = \frac{1}{2}(\varepsilon E^2 + \mu H^2)$$

In vacuum

$$u_{EM} = \frac{1}{2}(\varepsilon_0 E^2 + \mu_0 H^2)$$

Poynting Vector

The rate of energy transport per unit area in electromagnetic wave is described by a vector known as Poynting vector ($\vec{S}$) which is given as

$$\vec{S} = \vec{E} \times \vec{H} = \frac{\vec{E} \times \vec{B}}{\mu}$$

Poynting vector measures the flow of electromagnetic energy per unit time per unit area normal to the direction of wave propagation.

Unit of $\vec{S} \rightarrow 1 \frac{watt}{m^2}$ *in SI*.

Poynting Theorem

We have the Maxwell equations

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ----------(i)$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} ----------(ii)$$

Taking dot product $\vec{H}$ with (i) and $\vec{E}$ with (ii) and subtracting

$$\vec{H}.\vec{\nabla} \times \vec{E} - E.\vec{\nabla} \times \vec{H} = -\vec{H}.\frac{\partial \vec{B}}{\partial t} - \vec{E}.\frac{\partial \vec{D}}{\partial t} - \vec{E}.\vec{J} -------(iii)$$

$$LHS = \vec{\nabla}.(\vec{E} \times \vec{H})$$

$$\vec{H}.\frac{\partial \vec{B}}{\partial t} = \vec{H}.\frac{\partial(\mu \vec{H})}{\partial t} = \frac{\partial}{\partial t}(\frac{\mu H^2}{2})$$

Similarly

$$\vec{E}.\frac{\partial \vec{D}}{\partial t} = \vec{E}.\frac{\partial(\varepsilon \vec{E})}{\partial t} = \frac{\partial}{\partial t}(\frac{\varepsilon E^2}{2})$$

Then from (iii)

$$\vec{\nabla}.(\vec{E}\times\vec{H}) = -\frac{\partial}{\partial t}(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}) - \vec{E}.\vec{J}$$

$$\Rightarrow \vec{\nabla}.\vec{S} = -\frac{\partial u_{EM}}{\partial t} - \vec{E}.\vec{J} \qquad \text{as } \vec{E}\times\vec{H} = \vec{S} \quad and \quad u_{EM} = \frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}$$

This is sometimes called differential form of Poynting theorem.

Taking the volume integral of above

$$\int_V \vec{\nabla}.\vec{S}\, dV = -\int_V \frac{\partial u_{EM}}{\partial t} dV - \int_V \vec{E}.\vec{J}\, dV$$

Using Gauss divergence theorem to LHS we have

$$\oint_A \vec{S}.d\vec{A} = -\int_V \frac{\partial u_{EM}}{\partial t} dV - \int_V \vec{E}.\vec{J}\, dV$$

This represents Poynting theorem.

LHS of the equation $\rightarrow$ rate of flow of electromagnetic energy through the closed area enclosing the

given volume

1st term of RHS $\rightarrow$ rate of change of electromagnetic energy in volume

1st term of RHS $\rightarrow$ work done by the electromagnetic field on the source of current.

Thus Poynting theorem is a statement of conservation of energy in electromagnetic field.

In absence of any source, J=o

$$then\ \vec{\nabla}.\vec{S}+\frac{\partial u_{EM}}{\partial t}=0$$

This is called equation of continuity of electromagnetic wave.

Poynting Vector & Intensity of electromagnetic wave

$$Since\ \vec{E}\ and\ \vec{H}\ are\ mutually\ perpendicular$$

$$\left|\vec{S}\right| = EH = \frac{EB}{\mu}$$

Here E and H are instantaneous values.

$$Since\ \vec{E}\ and\ \vec{H}\ are\ in\ phase$$

$$\frac{E}{H}=\frac{E_0}{H_0}=\mu c$$

$$or\quad S=\frac{E^2}{\mu c}$$

$$If\quad E=E_0\sin\omega t,\ then\ average\ value\ of\ Poynting\ vector\ is$$

$$\langle S\rangle=\frac{\langle E_0^2\sin^2\omega t\rangle}{\mu c}=\frac{E_0^2}{2\mu c}\quad as\langle\sin^2\omega t\rangle=\frac{1}{2}$$

$$\Rightarrow\quad\langle S\rangle=\frac{c\varepsilon E_0^2}{2}=c\varepsilon E_{rms}^2\qquad as\ E_{rms}=\frac{E_0}{\sqrt{2}}$$

The average value of Poynting vector is the intensity (I) of the electromagnetic wave,

$$I=\langle S\rangle=c\varepsilon E_{rms}^2$$

QUANTUM PHYSICS

Need for quantum physics: Historical overview

- About the end of 19^{th} century, classical physics had attained near perfection and successfully explains most of the observed physical phenomenon like motion of particles, rigid bodies, fluid dynamics etc under the influence of appropriate forces and leads to conclusion that there is no more development at conceptual level.

- But some new phenomenon observed during the last decade of 19th century which is not explained by classical physics. Thus to explain their phenomena a new revolutionary concept was born which is known as Quantum physics developed by many outstanding physicists such as Planck, Einstein, Bohr, De Broglie, Heisenberg, Schrodinger, Born, Dirac and others.
- The quantum idea was 1st introduced by Max Planck in 1900 to explain the observed energy distribution in the spectrum of black body radiation which is later used successfully by Einstein to explain Photoelectric Effect.
- Neils Bohr used a similar quantum concept to formulate a model for H-atom and explain the observed spectra successfully.
- The concept of dual nature of radiation was extended to Louis De Broglie who suggested that particles should have wave nature under certain circumstances. Thus the wave particle duality is regarded as basic ingredient of nature.
- The concept of Uncertainty Principle was introduced by Heisenberg which explains that all the physical properties of a system cannot even in principle, be determined simultaneously with unlimited accuracy.
- In classical physics, any system can be described in any deterministic way where as in quantum physics it is described by probabilistic description.
- Every system is characterized by a wave function ψ which describes the state of the system completely and developed by Max Born.
- The wave function satisfies a partial differential equation called Schrodinger equation formulated by Heisenberg.
- The relativistic quantum mechanics was formulated by P.A.M. Dirac to incorporate the effect of special theory of relativity in quantum mechanics.

In this way, this leads to the development of quantum field theory which successfully describes the interaction of radiation with matter and describes most of the phenomena in Atomic physics, nuclear physics, Particle physics, Solid state physics and Astrophysics.

The Quantum Physics deals with microscopic phenomena where as the classical physics deals with macroscopic bodies. All the laws of quantum physics reduces to the laws of classical physics under certain circumstances of quantum physics are a super set then classical physics is a subset.

i.e., $\lim_{n\to 0} Quantum\ physics = Classical\ physics$

$$\lim_{n\to\infty} Classical\ physics = Quantum\ physics$$

PARTICLE ASPECTS OF RADIATION

The particle nature of radiation includes/are exhibited in the phenomena of black body radiation, Photoelectric effect, Compton scattering and pair production.

BLACK BODY RADIATION

- A black body is one which absorbs all them radiations incident on it.
- The radiations emitted by black body is called black body radiation.
- The black body emits radiation when it is heated at a fixed temperature and it contains all frequencies ranging from zero to infinity.
- The distribution of radiant energy among the various frequencies components of the black body radiation depends on its temperature.

  The energy distribution curve for black body radiation shows the following characteristics such as

  - ✓ At a given temperature the energy density has maximum value corresponding to a value of frequency or wavelength.

- ✓ The frequency corresponds to maximum energy density increases with increase of temperature.
- ✓ The energy density decreases to zero for both higher and lower values of frequency or wavelength.
- ✓ The energy density corresponding to a given frequency or wavelength increases with increase of temperature.

Many formulations are formulated to explain the above experimental observations like Stefan-Boltzmann law, Wein's displacement law and Planck's radiation formula. Out of which Planck's radiation formula successfully explains the facts of black body radiation.

PLANCK'S RADIATION FORMULA

According to Planck the black body was assumed to be cavity which consists of a large no. of oscillations with frequency ν and the empirical formula for energy distribution in the spectrum of black body radiation is given as

$$u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}d\nu \qquad (1)$$

$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}d\lambda \qquad (2)$$

In low frequency,

Lim(ν→ 0) $\quad [\nu= \frac{c}{\lambda}, d\nu = \frac{-c}{\lambda^2}d\lambda = \frac{c}{\lambda^2}d\lambda]$

i.e ν→ 0

$$e^{h\nu/kT} = 1 + \frac{h\nu}{kT}$$

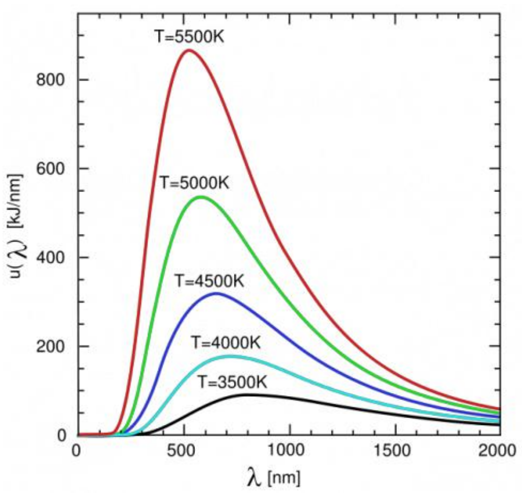
Therefore, $u(\nu)d\nu = \frac{8\pi\nu^3}{c^3}\frac{1}{1+\frac{h\nu}{kT}-1}$

$$u(\nu)d\nu=\frac{8\pi\nu^2}{c^3}kTd\nu$$

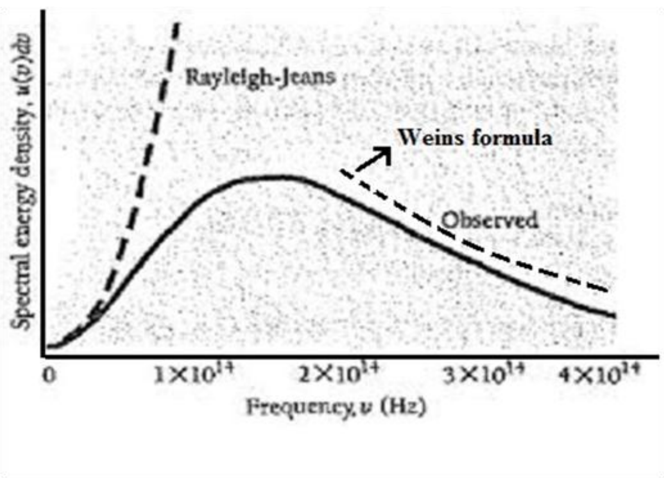
which is called Rayleigh-Jeans law.

In high frequency lim($\nu\rightarrow\infty$)

i.e $\nu \rightarrow \infty, \frac{h\nu}{kT} \rightarrow \infty, e^{h\nu/kT} - 1 \cong e^{h\nu/kT}$



Therefore, $u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}e^{-h\nu/kT}d\nu$, which is called Wein's radiation formula.

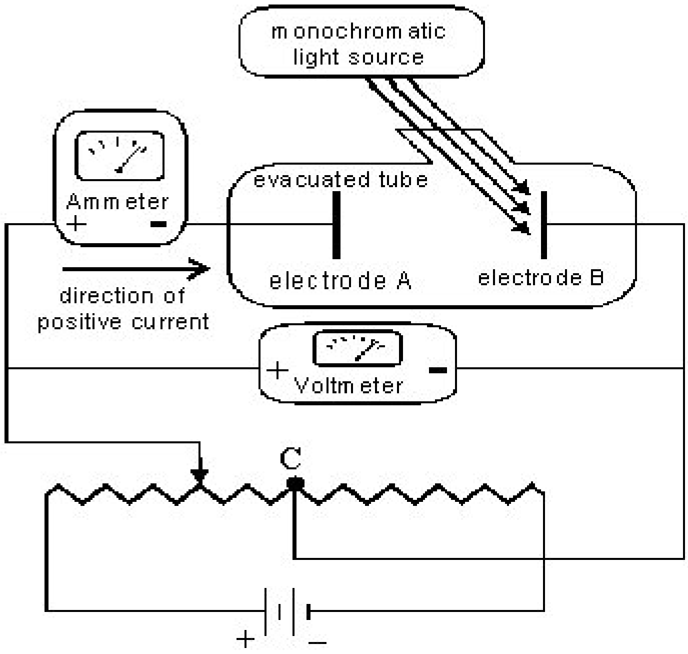


PHOTOELECTRIC EFFECT

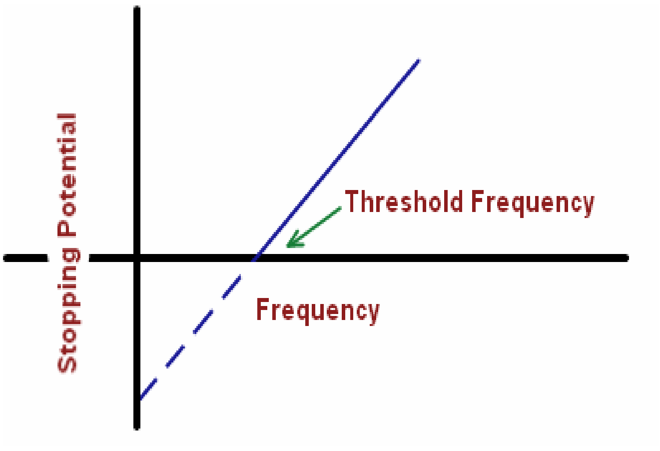
The phenomenon of emission of electron from surface of certain substance when a light of suitable frequency or wavelength incident on it is called Photoelectric effect.

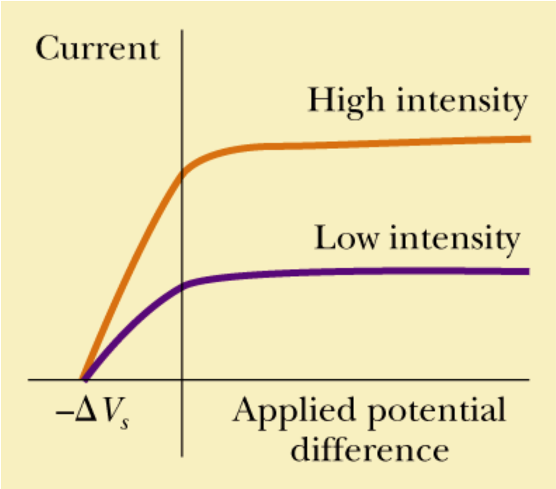
Experimental Arrangement

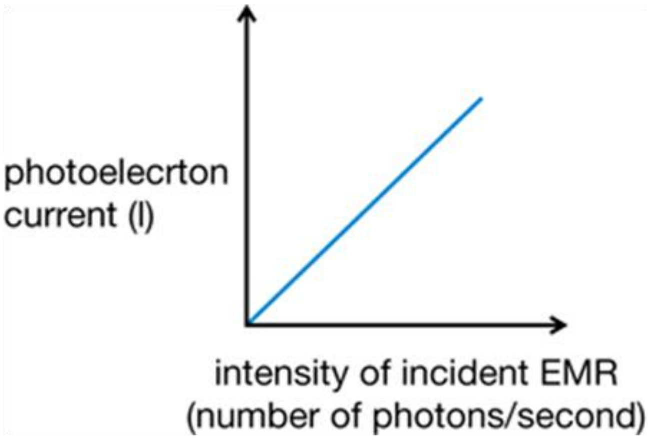
The experimental arrangement consists of the following parts.

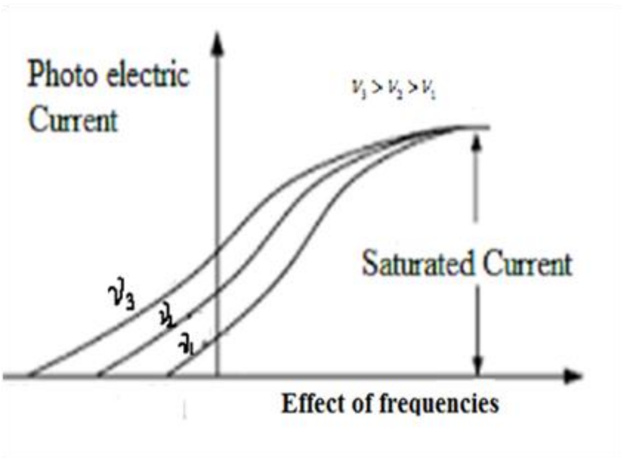


Experimental results are represented graphically as follows









<u>Laws of Photoelectric effect</u>

- It is an instantaneous process.
- It is directly proportional to intensity of incident light.
- Photocurrent is independent of frequency of incident light.

- Stopping potential depends upon the frequency but independent of intensity.
- The emission of electrons stops below certain minimum frequency called threshold frequency.
- Saturation current is independent of frequency.

Einstein's Theory of Photoelectric Effect

According to Einstein, when light of frequency ν is incident on a metallic surface, each photon interacts with one electron and completely transfers its energy to the electron, this energy is utilized in two ways.

i. A part of this energy is used to the free electron from the atom and away from the metal surface n overcoming the work function(W_0).

ii. The other part is used in giving K.E to the electron ($\frac{1}{2}mv^2$).

Thus according to the law of conservation of energy,

$$h\nu = \frac{1}{2}mv^2 + W_0$$

(1)

of the frequency of the incident light ν_0 is required to remove the electron, then

$$h\nu_0 = W_0 \qquad (2)$$

using eq^n (2) in eq^n(1), we get

$$h\nu = \frac{1}{2}mv^2 + h\nu_0$$

$$h(\nu - \nu_0) = \frac{1}{2}mv^2$$

(3)

<u>Calculation of stopping Potential(ν_0)</u>

To neutralize the K.E of the emitted electron, we have

$$\frac{1}{2}mv^2 = eV_0$$

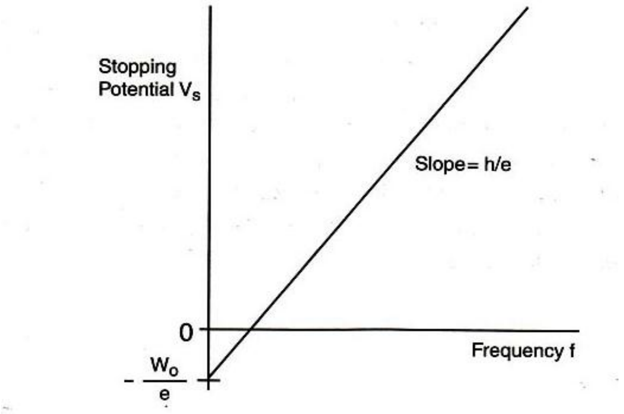
(4)

Using eq^n (4) in eq^n(1), we get

$$h\nu = eV_0 + W_0$$

implies that $$\nu_0 = \frac{h}{e}\nu - \frac{W_0}{e}$$

(5)



<u>Calculation of threshold frequency</u>

We have

$$\nu_0 = \frac{h}{e}\nu - \frac{W_0}{e}$$

We have

$$\frac{W_0}{e/h} = \frac{W_0}{eh}$$

(6)

Substituting eq^n(2) in eq^n(6) we get.

$$\frac{Y - intercept}{h} = \frac{h\nu_0}{h} = \nu_0$$

$$\Rightarrow \boxed{\nu_0 = \frac{y - \text{int}\,ercept\,in\,\nu_0}{h}}$$

<u>Calculation of work function</u>

From the plot of $\nu_0 \sim \nu$ we have slope of the done is $\frac{-W_0}{e}$ if we multiply 'e' with the intercept, we get

$$\frac{W_0}{e} \times e = W_0$$

i.e $$W_0 = y - intercept\ in\ \nu_0 \sim \nu \times e$$

<u>Calculation of Planck Constant (h)</u>

If we multiply the slope of plot of stopping potential $\nu_0 \sim \nu$ with 'e' we get 'h'

i.e $$h = slope\ of\ \nu_0 \sim \nu \times e$$

$$h = \frac{h}{e} \times e$$

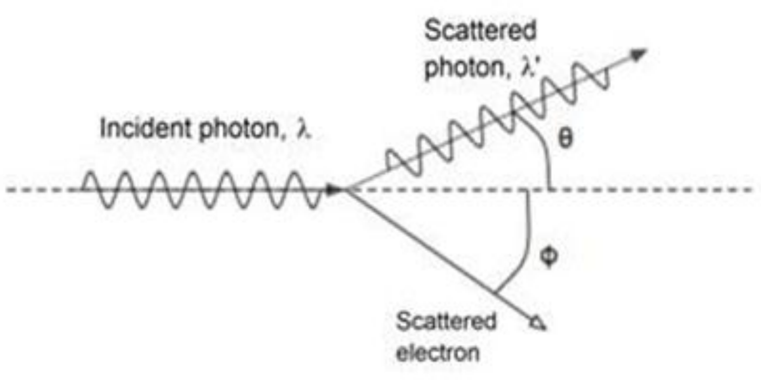
Q. Is wave nature of radiation successfully explains the Compton effect? Justify your answer.

Ans. No

<u>Compton effect</u>

The phenomena in which a beam of high frequency radiation like x-ray &γ-ray is incident on a metallic block and undergoes scattering is called Compton effect.

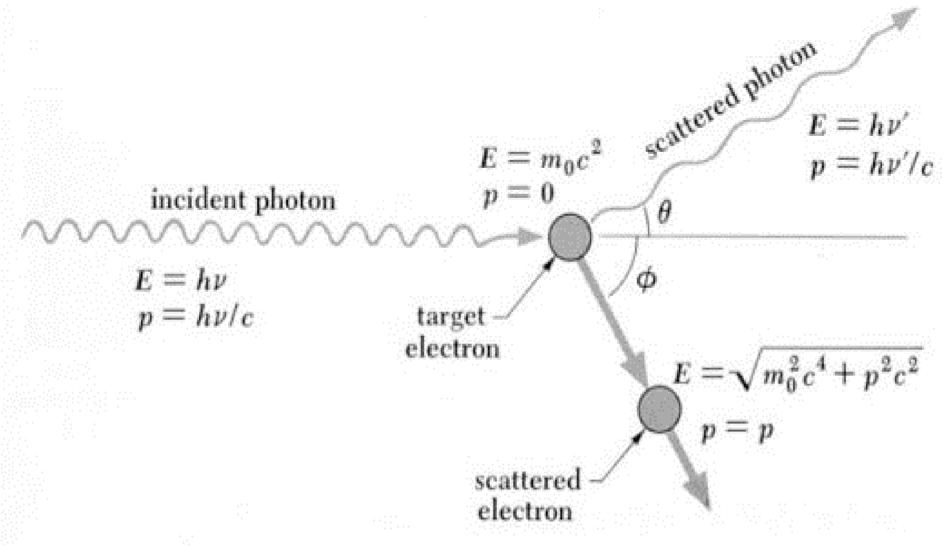
- The component whose wavelength is same as that of incident radiation is called unmodified line (Thomson component)
- The component whose wavelength is greater than the incident wavelength is called modified line (Compton component)
- The increase of wavelength in the Compton component is called Compton Shift (Δλ).
- It depends on the angle of scattering (angle between the scattered & incident x-ray).
- It is independent of the wavelength of the incident x-ray.



*Wave nature of radiation is unable to explain Compton shift as the Compton shift depends on angle of scattering and wavelength of scattered x-ray is different from that of the incident x-rays.

<u>Comptons Explanation:-</u>

Let us consider a photon of energy hν collide with an electron which is at rest. Though the electron is closely bound with the nucleus, but a small fraction of energy is used to free the electron. During the collision the photon gives a fraction of energy to the free electron and the electron gain K.E and recoils at an angle ϕ to the incident photon direction after collision and the photon with decrease energy hν' will emerges at an angle θ to the initial direction after collision.



Applying law of conservation of energy, we have

$$h\nu + m_0c^2 = h\nu' + mc^2 \qquad (1)$$

The relativistic variation of mass is given as

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}} \qquad (2)$$

According to law of conservation of momentum,

$$\frac{h\nu}{c} + 0 = \frac{h\nu'}{c}cos\theta + mvcos\phi \qquad (3)$$

$$=> 0 + 0 = \frac{h\nu'}{c}\sin\theta - mv\sin\phi$$

(4)

From equation (3)

$$\text{mvccos}\,\phi = h\nu - h\nu' \cos\theta$$

(5)

and from equation (4)

$$mvc \sin\phi = h\nu' \sin\theta$$

(6)

Squaring and adding equation (5) and (6) we get

$$m^2v^2c^2 = (h\nu - h\nu' \cos\theta)^2 + (h\nu' \sin\theta)^2$$

$$=h^2\nu^2 - 2h^2\nu\nu' \cos\theta + h^2\,\nu'^2 cos^2\theta +$$

$$h^2\nu'^2 sin^2\theta$$

$$= h^2\nu^2 - 2h^2\nu\nu' cos\theta + h^2\nu'^2$$

$$=> m^2v^2c^2 = h^2(\nu^2 + {\nu'}^2 - 2\nu\nu' cos\theta) \qquad (7)$$

From eq^n(1), we get

$$mc^2 = h(\nu - \nu') + m_0c^2$$

$$=> m^2c^4 = h^2\left(\nu^2 - 2\nu\nu' + {\nu'}^2\right) + 2h(\nu - \nu')m_0c^2 + m_0^2c^4 \qquad (8)$$

Subtracting eq^n(7) from eq^n(8)

$$m^2c^4 - m^2v^2c^2 = -2h^2\nu\nu'^{(1-cos\theta)} + 2h(\nu - \nu')m_0c^2 + m_0^2c^4$$

$$=> m^2c^2(c^2 - v^2)$$

$$= -2h^2\nu\nu'(1 - cos\theta) + 2h(\nu - \nu')m_0c^2 + m_o^2c^4$$

$$\Rightarrow \frac{m_0^2c^2}{1-{v^2}/{c^2}}\left(c^2 - v^2\right) = -2h^2\nu\nu'(1-\cos\theta) + 2h\left(\nu - \nu'\right)m_0c^2 + m_0^2c^4$$

$$\Rightarrow m_0^2c^4 = -2h^2\nu\nu'\left(1-\cos\theta\right) + 2h\left(\nu - \nu'\right)m_0c^2 + m_0^2c^4$$

$$\Rightarrow 2h\left(\nu - \nu'\right)m_0c^2 = 2h^2\nu\nu'\left(1-\cos\theta\right)$$

$$\Rightarrow \frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{m_0 c}(1-\cos\theta)$$

$$\Rightarrow \lambda' - \lambda = \frac{h}{m_0 c}(1-\cos\theta)$$

$$\boxed{\Rightarrow \Delta\lambda = \lambda_c (1-\cos\theta)}$$

Where $\Delta\lambda = \lambda' - \lambda$

=Compton shift

$\lambda_c = \frac{h}{m_0 c}$ =Compton wavelength

$$= \frac{6.62\times10^{-34}\,JS}{\left(9.11\times10^{-31}kg\right)\times\left(3\times10^{8}\,{}^{m}\!/\!_{s}\right)}$$

$$= 2.426\times10^{-12}m$$

$$\boxed{\lambda_c = 0.0242A^0}$$

$\lambda_c =$ has dimension of length

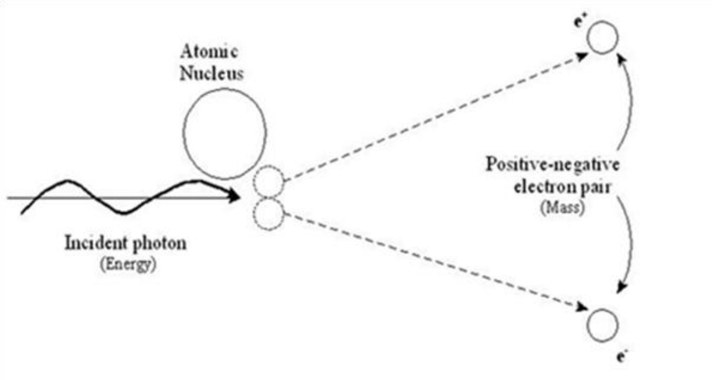
* The Compton wavelength for any other particles is $\frac{h}{mc}$
* When $\theta = 0, \boxed{\Delta\lambda = 0} \Rightarrow$ There is no scattering or Compton shift along the incident direction.

  $\theta = {}^{\pi}\!/\!_{2}, \boxed{\Delta\lambda = \lambda_c}$

  $\theta = \pi, \Delta\lambda = 2\lambda_c$ (Maximum shift)

<u>Pair Production:</u>

The phenomenon in which some γ-rays are converted into electron-positron pair on passing near an atomic nucleus is called Pair production.



Pair Production - Energy Converstion to Mass

- It is an example of conservation of energy and momentum in the nature.
- Pair production is not possible if the γ-rays are treated as EM waves for which the pair production is not possible in vacuum.
- For pair production the suitable condition is $h\nu \geq 2m_e c^2$.
- The minimum frequency ν_0 of γ-rays for which $h\nu_0 = 2m_e c^2$ and the pair production takes place is called threshold frequency.

* Pair production takes place for high frequency EM wave. (γ-ray)
Compton effect takes place for intermediate frequency value. (x-ray)
Photoelectric effect takes place for frequency corresponding to UV-waves.

Matter waves and De-Broglie Hypothesis

The waves associated with all material particles are called Matter waves.

According to De-Broglie hypothesis, the wavelength λ of matter wave associated with a moving particle of linear momentum P is given by

$$\boxed{\lambda = \frac{h}{P}}$$

or, $$\boxed{\lambda = \frac{h}{mv}}$$

For a non-relativistic free particle of kinetic energy E, we have

$$E = \frac{P^2}{2m}$$
$$\Rightarrow P = \sqrt{2mE}$$
$$\therefore \lambda = \frac{h}{\sqrt{2mE}}$$

If q=charge of a particle

m=mass of the particle

V=potential difference

Then, $\frac{P^2}{2m} = qv \Rightarrow P = \sqrt{2mqv}$

$$\therefore \boxed{\lambda = \frac{h}{\sqrt{2mqv}}}$$

If T=absolute temperature, then

$$\frac{P^2}{2m} = \frac{3}{2}kT \Rightarrow P = \sqrt{3mkT}$$
$$\boxed{\Rightarrow \lambda = \frac{h}{\sqrt{3mkT}}}$$

For a free relativistic particle,

$$E = \sqrt{P^2c^2 + m_0^2c^4}$$

$$\Rightarrow P = \frac{\sqrt{E^2 - m_0^2c^4}}{c}$$

$$\therefore \boxed{\lambda = \frac{hc}{\sqrt{E^2 - m_0^2c^4}}}$$

* Experimental confirmation of matter wave was demonstrated by Davision-Germer experiment.
* The wave nature of electron was demonstrated by division and Germer.

Heisenberg's Uncertainty Principle:

It states that it is impossible to measure simultaneously the position and the corresponding component of its linear momentum with unlimited accuracy.

If Δx= uncertainty in x-component of the position of a particle

Δp_x= uncertainty in x-component of its linear momentum

then, $\Delta x \cdot \Delta p_x \geq \frac{\hbar}{2}$

Similarly for y and z-component

$\Delta y \cdot \Delta p_y \geq \frac{\hbar}{2}$, $\Delta z \cdot \Delta p_z \geq \frac{\hbar}{2}$

Again uncertainty in energy and time is given by

$\Delta t \cdot \Delta E \geq \frac{\hbar}{2}$

Application of the uncertainty principle;

i. Ground state energy of harmonic oscillator

The energy of the 1-D harmonic oscillator is given as

$$E = \frac{P^2}{2m} + \frac{1}{2}m\omega^2x^2 \qquad (1)$$

Let us assume that in the ground state, the linear momentum P and position x of the oscillator are of the order of their uncertainties.

i.e $\Delta P \approx P$ and $\Delta x \approx x$

According to principle

$$\Delta x.\Delta p_x \geq \frac{\hbar}{2}$$
$$\Rightarrow \Delta x.\Delta p_x \approx \hbar$$
$$\Rightarrow p.x \approx \hbar$$

$$\Rightarrow p \approx \frac{\hbar}{2} \qquad (2)$$

Using eq^n(2) in eq^n(1),we get

$$E = \frac{\hbar^2}{2mx^2} + \frac{1}{2}m\omega^2 x^2 \qquad (3)$$

Since the energy E of the oscillator is minimum in the ground state, so

$$\left(\frac{\partial E}{\partial x}\right)_{x=x_0} = 0$$

$$\Rightarrow 0 = \left(\frac{\partial E}{\partial x}\right)_{x=x_0} = \frac{-\hbar^2}{mx_0^3} + m\omega^2 x_0$$

$$\Rightarrow x_0^2 = \frac{\hbar}{m\omega} \qquad (4)$$

Where x_0 corresponds to the ground state.

Using eq^n (4) in eq^n (3), we get

$$E_0 = \frac{\hbar^2}{2m\left(\frac{\hbar}{m\omega}\right)} + \frac{1}{2}m\omega^2 \frac{\hbar}{m\omega} = \hbar\omega \qquad (5)$$

Thus the minimum energy of 1-D harmonic oscillator cannot be zero.

ii. <u>Non-existence of electron in the nucleus</u>

Let us assume that the electron is inside the nucleus whose uncertainty in position is given as

$$\Delta x \approx 10^{-14} m$$

From the principle

$$\Delta x.\Delta p \approx \frac{\hbar}{2}$$

$$\Rightarrow p \approx \Delta p \approx \frac{\hbar}{2\Delta x} \approx \frac{6.62\times10^{-34}\,Js}{2\times2\times3.14\times10^{-14}\,m} = 5.3\times10^{-21}\,\frac{kg.m}{s}$$

The minimum energy of the electron in the nucleus is

$$E = \sqrt{m_0^2c^4 + p^2c^2}$$

$$= \left[\left(9.1\times10^{-31}kg\right)^2\left(3\times10^8\,{}^{m}\!/_{s}\right)^4 + \left(5.3\times10^{-21}\,{}^{kg.m}\!/_{s}\right)^2\times\left(3\times10^8\,{}^{m}\!/_{s}\right)^2\right]^{1/2}$$

$$E = 1.6\times10^{-12}\,J = 10MeV$$

As energy of the electrons emitted in β-decay process is much less than this estimated value, so the electrons cannot be a part of the nucleus.

iii. <u>Ground State energy of the H-Atom</u>

The energy of the H-atom is given as

$$E = \frac{p^2}{2m} - \frac{e^2}{4\pi\in_0 r} \qquad (1)$$

Let $\Delta r \rightarrow$Uncertainty in position of the electron in the orbit of radius r in the ground state.

$\Delta p \rightarrow$Uncertainty in momentum of the electron in the ground state.

Then using principle,

$$\Delta r.\Delta p \geq \frac{\hbar}{2}$$

$$\Rightarrow \Delta r.\Delta p \approx \hbar$$

$$\Rightarrow r.p \approx \hbar$$

$$\Rightarrow p \approx \frac{\hbar}{r} \qquad (2)$$

Now using eq^n (2) in eq^n (1), we get

$$E = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi \in_0 r} \qquad (3)$$

As the ground state energy E_0 is minimum, so

$$\left(\frac{\partial E}{\partial r}\right)_{r=o} = 0$$

$$\Rightarrow \left[\frac{-\hbar^2}{mr^3} + \frac{e^2}{4\pi \in_0 r^2}\right]_{r=r_0} = 0$$

$$\Rightarrow r_0 = \frac{4\pi \in_0 \hbar^2}{me^2}$$

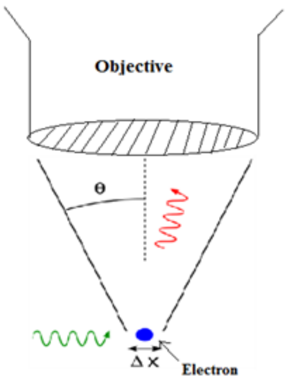
Thus eq^n (3) becomes,

$$\boxed{E_0 = \frac{me^4}{32\pi^2 \in_0^2 \hbar^2}}$$

Examples:

1) Heisenberg's gamma-rays Microscope:
If Δx be the uncertainty in position of the electron decided by Resolving power of the microscope, then

$$\Delta x = R.P = \frac{1.22\lambda}{2\sin\theta}$$



As the scattered γ-ray photon enter the objective of the microscope, so its linear momentum can be resolved into component along its x-axis which is given as,

$$\Delta p_x = \frac{2h}{\lambda}\sin\theta$$

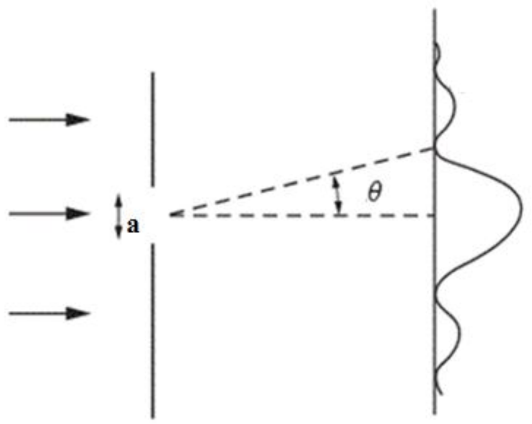
According to principle,

$$\Delta x.\Delta p_x = \frac{1.22\lambda}{2\sin\theta}\times\frac{2h}{\lambda}\sin\theta = 1.22h$$
$$\Rightarrow \boxed{\Delta x.\Delta p_x \approx h}$$

Which agrees with the uncertainty relation.

2) <u>Electron Diffraction:</u>
Let a be the width of the slit through which the electron beam is diffracted along y-direction. Then diffraction condition is,

$$a\sin\theta = \lambda \qquad (1)$$



Let Δp be the uncertainty in momentum along y-direction, then

$$\Delta p_y = 2p\sin\theta \qquad (2)$$

Now multiplying eq^n(1) and eq^n(2), we get

$$\Delta y.\Delta p_y = a(2p\sin\theta) = 2p(a\sin\theta)$$

$$= 2p\lambda = 2p.\frac{h}{p} = 2h$$

$$\Rightarrow \boxed{\Delta y.\Delta p_y \geq \frac{\hbar}{2}}$$

which satisfies the uncertainty principle.

<u>Transition from deterministic to probabilistic</u>

In classical physics, the physical properties of a system can be specified exactly in principle. If the initial conditions of a system are known, its subsequent configurations can be determined by using the relevant laws of physics applicable to the system. Thus classical physics is deterministic in nature. But this deterministic description is inconsistent with observation. In quantum mechanics every physical system is characterized by a wave function which contains all the information's for the probabilistic description of a system. This probabilistic description is the basic characteristic of quantum physics and is achieved by the wave function.

<u>Wave function</u>

- The state function which contains all information's about a physical system is called wave function $\psi(\vec{r},t)$.
- It describes all information's like amplitude, frequency, wavelength etc.
- It is not a directly measurable quantity.
- It is a mathematical entity by which the observable physical properties of a system can be determined.

<u>Characteristics</u>

- It is a function of both space and time co-ordinate.
  i.e. $\psi(\vec{r},t) = \psi(x, y, z; t)$
- It is a complex function having both real and imaginary part.
- It is a single valued function of its arguments.

- The wave function ψ and its first derivative $\frac{\partial \psi}{\partial x}$ are continuous at all places including boundaries.
- It is a square integrable function i.e. $|\psi|^2\, dv = 1$.
- The quantity $|\psi|^2$ represents the probability density.
- It satisfies the Schrodinger's equation.

<u>Superposition principle</u>

This principle states that "Any well behaved state of a system can be expressed as a linear superposition of different possible allowed states in which the system can exists."

If ψ_1, ψ_2, ψ_3......be the wave functions representing the allowed states, then the state of the system can be expressed as

$$\psi = \psi_1 + \psi_2 + \psi_3 + ... + \psi_n = \sum c_n \psi_n$$

<u>Probability density</u>

The probability per unit volume of a system being in the state ψ is called probability density.

i.e. $\rho = |\psi|^2$

As the probability density is proportional to square of the wave function, so the wavefunction is called "probability amplitude".

The total probability is,

$$\int_v \rho dv = \int_v |\psi|^2 dv = 1$$

As the total probability is a dimensional quantity, so it has dimension $\left[L^{-3}\right]$ and the wavefunction has dimension $\left[L^{-3/2}\right]$.

- Dimension of 1-D wave function is $\left[L^{-1/2}\right]$.

- Dimension of 2-D wave function is $\left[L^{-1}\right]$.

Observables

The physical properties associated with the wave function provides the complete description of the system state or configuration are called observables.

Ex: energy, angular momentum, position etc.

Operators

The tools used for obtaining new function from a given function are called operators.

If $\hat{A}$ be an operator and f(x) be a function, then $\hat{A}f(x)=g(x)$; g(x)=new function

Ex: energy operator, momentum operator, velocity operator etc.

Physical Quantity	Operator
Energy-E	$i\hbar\frac{\partial}{\partial t}$
Momentum- $\vec{p}$	$-i\hbar\vec{\nabla}$
Potential Energy(V)	V
Kinetic Energy($p^2/2m$)	$\frac{-\hbar^2}{2m}\nabla^2$

Eigen States:

The number of definite allowed states for the system are called eigen states.

Eigen Values:

The set of allowed values of a physical quantity for a given system is called eigen values of the quantity.

For any operator $\hat{A}$ having eigen values α_i corresponding to the eigen states ψ_i the eigen value equation is $\boxed{\hat{A}\psi_i = \alpha_i\psi_i}$

Expectation Values:

The expectation values of a variable is the weighted average of the eigen values with their relative probabilities.

If $q_1, q_2, q_3, ...$ are the eigen values of a physical quantity Q and they occur with probabilities $p_1, p_2, p_3, ...$ for a given state of the system then weighted average of Q is

$$\langle Q \rangle = \frac{p_1 q_1 + p_2 q_2 + ...}{p_1 + p_2 + ...} = \frac{\sum p_n q_n}{\sum p_n}$$

Since the total probability is 1, so $p_1 + p_2 + p_3 + ... = 1$

$$\therefore \langle Q \rangle = p_1 q_1 + p_2 q_2 + ... = \sum p_n q_n$$

In general if A be a physical quantity, then

$$\langle A \rangle = \int \alpha |\psi|^2 \, dV$$

$$= \int \psi^* \alpha \psi \, dv$$

$$\boxed{\langle A \rangle = \int \psi^* \hat{A} \psi \, dV} \qquad \because \hat{A}\psi = \alpha\psi$$

For normalized wave function.

* For any function to be normalized is given as

$$\boxed{\int\left|\psi\left(\vec{r},t\right)\right|^2 dV = 1}$$

* The expectation value of energy,

$$\langle E\rangle = \int \psi^* \hat{H}\psi dV = \int \psi^* \left(i\hbar\frac{\partial}{\partial t}\right)\psi dV$$
$$= i\hbar\int \psi^* \frac{\partial \psi}{\partial t} dV$$

<u>Schrodinger's Equation:-</u>

The partial differential equation of a wave function involving the derivatives of space and time coordinates is called Schrodinger equation.

<u>Time-dependent Schrodinger equation</u>

Let the wave function be represented by

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

$$\left.\begin{aligned} &\Rightarrow \frac{\partial \psi}{\partial x} = ik\psi, \frac{\partial \psi}{\partial t} = -i\omega\psi \\ &\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -k^2\psi \end{aligned}\right\} \qquad (1)$$

The energy and momentum are given as

$$E = h\nu = \hbar\omega$$
$$p = \frac{h}{\lambda} = \hbar k$$

We have $\qquad E = \dfrac{p^2}{2m}$

$$\Rightarrow \hbar\omega = \frac{\hbar^2 k^2}{2m} \qquad (2)$$

Using eq^n(1) in eq^n(2),

$$\Rightarrow \boxed{i\hbar\frac{\partial\psi}{\partial t}=\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}} \qquad (3)$$

This is the time-dependent Schrodinger equation for a free particle in 1-dimension.

If the particle is in a potential V(x), then

$$E=\frac{p^2}{2m}+V$$

$$\Rightarrow \boxed{i\hbar\frac{\partial\psi}{\partial t}=\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V\psi} \qquad (4)$$

Similarly along Y and Z-axis is given as

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2}+V\psi$$

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2}+V\psi$$

<u>Time-dependent Schrodinger equation in 3-D:</u>

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi+V\psi$$

$$\boxed{i\hbar\frac{\partial\psi}{\partial t}=\frac{-\hbar^2}{2m}\nabla^2\psi+V\psi} \qquad (5)$$

<u>Time-independent Schrodinger equation:</u>

If the energy of the system does not change with time then

$E=\hbar\omega$ remains constant

Now from eq^n (1),

$$i\hbar\frac{\partial\psi}{\partial t}=i\hbar\left(-i\omega\psi\right)=\hbar\omega\psi=E\psi$$

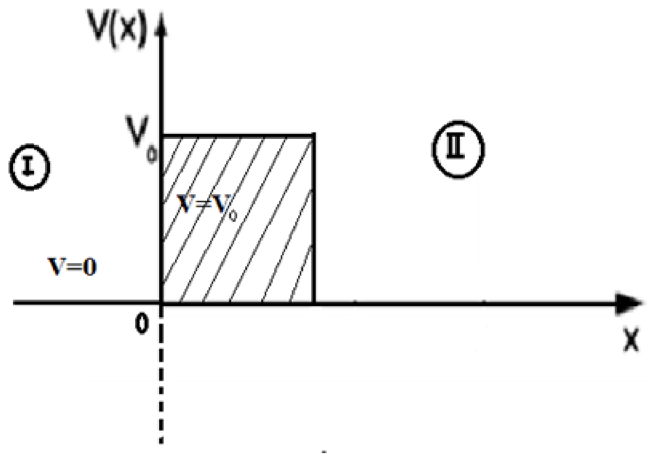
$$\therefore E\psi=\frac{-\hbar^2}{2m}\nabla^2\psi+V\psi \qquad \text{[fromeq}^n\text{ (5)]}$$

$$\Rightarrow \boxed{\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0}$$

This is time-independent Schrodinger equation in 3-D.

Potential step:

The physical situation in which the potential energy of a particle changes from one constant value (V_1) to another constant value (V_2) when the particle changes from one region to another is called potential step.



The potential step can be given as

$$V(x) = 0, x < 0$$

$$= V_0, x > 0$$

Let us consider the particle incident on the potential step from left to right. According to the classical physics if the particle has energy less than the potential step, the particle cannot move beyond x=0 and will rebound into region-1. If the energy of the particle is greater than the height of the potential step the particle will go to the region-2.

Case-1:($E>V_0$)

Let ψ_1 and ψ_2 be the wave function in region-1 and region-2 then time dependent Schrodinger equation is given as

$$\frac{d^2\psi_1}{dx^2}+\frac{2m}{\hbar^2}\psi_1=0$$, x<0, (region-1)

(1)

$$\frac{d^2\psi_2}{dx^2}+\frac{2m}{\hbar^2}(E-V_0)\psi_2=0$$,x>0,(region-2)

(2)

$$\Rightarrow\frac{d^2\psi_1}{dx^2}+k_1^2\psi_1=0 \quad (3)$$

And $$\frac{d^2\psi_2}{dx^2}+k_2^2\psi_2=0 \quad (4)$$

Where $$k_1=\frac{2mE}{\hbar^2} \quad (5)$$

$$k_2=\frac{2m}{\hbar^2}(E-V) \quad (6)$$

The general solution of eq^n(3) and eq^n(4) is given as

$$\psi_1(x)=Ae^{ik_1x}+Be^{-ik_1x} \quad (7)$$

$$\psi_2(x)=Ce^{ik_2x}+De^{-ik_2x} \quad (8)$$

Where Ae^{ikx} -incident wave= ψ_{inc}

Be^{-ikx} -reflected wave= ψ_{ref}

Ce^{ik_2x} -transmitted wave= ψ_{tran}

De^{-ik_2x} -wave incident from right to left in region-2 for which it is zero.

Thuseq^n (8) becomes

$$\psi_2(x)=Ce^{ik_2x} \quad (9)$$

Using boundary condition,

$$\psi_1(x)\big|_{x=0} = \psi_2(x)\big|_{x=0}$$
$$\frac{\partial \psi_1}{\partial x}\bigg|_{x=0} = \frac{\partial \psi_2}{\partial x}\bigg|_{x=0}$$

We have A+B=C and $ik_1(A-B) = ik_2 e$

$$\Rightarrow B = \frac{k_1 - k_2}{k_1 + k_2} A$$

(10)

Thus it is observed that

1. R+T=1, which follows from the conservation of flux.
2. It explains wave nature of particles by the fact that the probability of particle is not zero in the region-2 which iscontradictory to classical physics.
3. If barrier height V_0<E(incident energy) then incident particle do not see the potential step and are almost transmitted as per the classical physics.
4. If $V_0 \approx E$, then the quantum effect become prominent and the reflection is appreciable.

<u>Case-2:</u> ($E<V_0$)

Now the Schrodinger equation in region-2 is given as

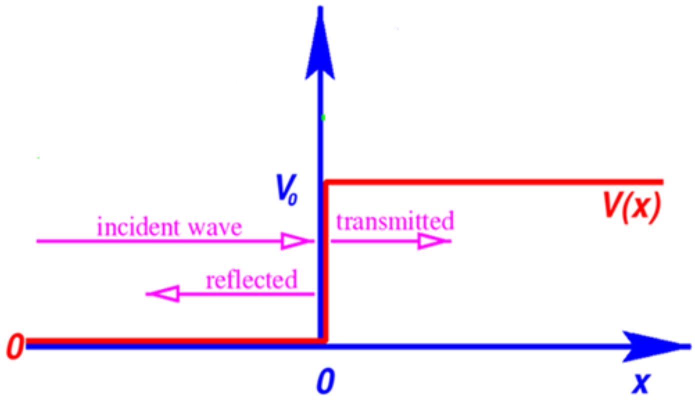
$$\frac{d^2\psi_2}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)\psi_2 = 0$$
$$\Rightarrow \frac{d^2\psi_2}{dx^2} - \alpha^2\psi_2 = 0 \qquad (1)$$

Where $\alpha = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$

Thus the solution of the equation is given as

$$\psi_2(x) = Ce^{-\alpha x} + De^{\alpha x}$$

And $C = \frac{2k_1}{k_1 + k_2} A$



Reflection Coefficient

It is defined as the ratio of reflected flux to incident flux of the particle.

i.e $R = \frac{V_1 |\psi_{ref}|^2}{V_1 |\psi_{inc}|^2}, V_1 = \frac{\hbar k_1}{m}$

$$= \frac{\psi^*_{ref}\psi_{ref}}{\psi^*_{inc}\psi_{inc}} = \frac{|B|^2}{|A|^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

$$\boxed{R = \frac{\left(\sqrt{E} - \sqrt{E - V_0}\right)^2}{\left(\sqrt{E} + \sqrt{E - V_0}\right)^2}}$$

Transmission Coefficient:-

It is defined as the ratio between transmitted flux to incident flux.

i.e $T = \frac{\text{Transmittedflux}}{\text{Incidentflux}}$

$$= \frac{V_2 |\psi_{tran}|^2}{V_1 |\psi_{inc}|^2} \quad , V_2 = \frac{\hbar k_2}{m}$$

$$= \frac{k_2 \psi^*_{tran}\psi_{tran}}{k_1 \psi^*_{inc}\psi_{inc}} = \frac{k_2 |C|^2}{k_1 |A|^2}$$

$$\boxed{T = \frac{4k_1k_2}{(k_1 + k_2)^2} = \frac{4\sqrt{E}\sqrt{E - V_0}}{\left(\sqrt{E} + \sqrt{E - V_0}\right)^2}}$$

As $x \to \infty, \psi \to 0 \Rightarrow De^{\alpha x} \to \infty$,for which D=0

So $\psi_2(x) = Ce^{-\alpha x}$ (2)

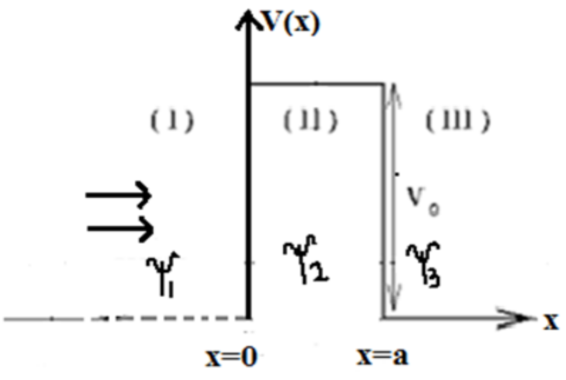
Which indicates that the probability of finding of particle in region-2 is not zero which is classically forbidden as there is some particles on region-2 according to quantum mechanics.

Potential barrier:-

The physical situation in which the potential of a region varies between zero and maximum outside and inside the confined region is known as potential barrier.

The potential of such region is given by, V(x) =0, x<0 and x>a

$=V_0$, $0 \le x \le a$



Let us consider a particle is travelling from left to right. As per the classical physics the particle cannot cross the barrier if $E<V_0$ but according to quantum mechanics there is non-zero probability of the particle of crossing the barrier even if $E<V_0$.

Case-1($E<V_0$)

Let ψ_1 be the wave function of the particle describes the motion of the particles in the region-1, then

$$\frac{d^2\psi_1}{dx^2}+\frac{2mE}{\hbar^2}\psi_1=0$$

$$\Rightarrow \frac{d^2\psi_1}{dx^2}+k^2\psi_1=0 \qquad (1)$$

where $k=\sqrt{\frac{2mE}{\hbar^2}}$

In region-2, Schrodinger wave equation is given as

$$\frac{d^2\psi_2}{dx^2}-\frac{2m}{\hbar^2}(V_0-E)\psi_2=0$$

$$\Rightarrow \frac{d^2\psi_2}{dx^2}-\alpha^2\psi_2=0 \qquad (2)$$

Where $\alpha=\sqrt{\frac{2m}{\hbar^2}(V_0-E)}$

In region-3,

$$\frac{d^2\psi_3}{dx^2}+\frac{2m}{\hbar^2}\psi_3=0$$

$$\Rightarrow \frac{d^2\psi_3}{dx^2}+k^2\psi_3=0 \qquad (3)$$

The general solution of Schrodinger equation in the three region is given by

$$\psi_1(x)=Ae^{ikx}+Be^{-ikx} \qquad (4)$$

$$\psi_2(x)=Ce^{\alpha x}+De^{-\alpha x} \qquad (5)$$

$$\psi_3(x)=Fe^{ikx}+Ge^{-ikx} \qquad (6)$$

Where $Ae^{ikx}, Be^{ikx} \longrightarrow$ the incident and reflected waves in region-1

$Fe^{ikx} \longrightarrow$ Transmitted wave in region-2

Ge^{-ikx} →wave incident from right in region-3

=0

$$\therefore \psi_3 = Fe^{ikx} \qquad (7)$$

The wave function ψ_1 and ψ_2 and their derivatives continuous at x=0. Similarly ψ_2 and ψ_3 and their derivatives should be continuous at x=a. But in the region-2 the wave function is non-zero. Thus at the boundary x=a, continuity wave function $\psi_2 = \psi_3$ requires that ψ_3 is non-zero at region-3. Thus there is non-zero probability of finding the particles in region-3 even if the incident particle energy is less than the barrier height.

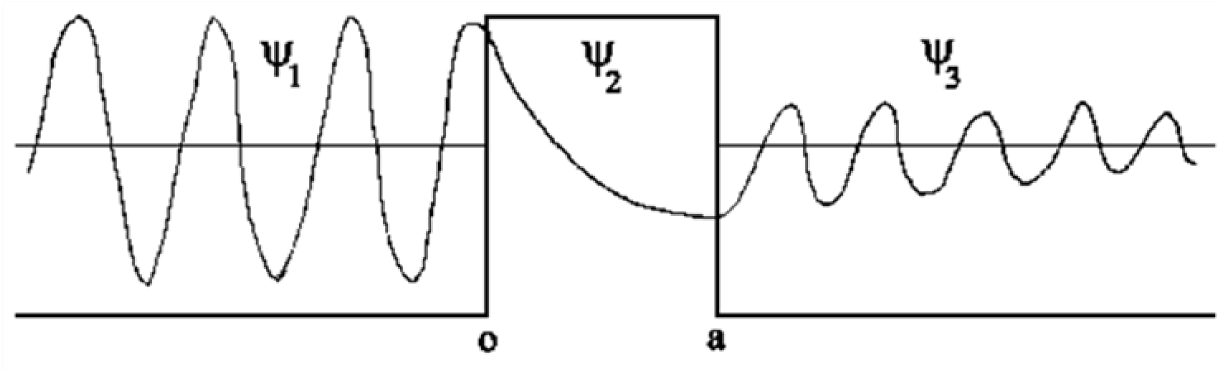
<u>Quantum mechanical tunneling:</u>

The phenomena in which the particles penetrate through the barrier is called quantum mechanical tunneling.

Ex: Emission of α-particle, nuclear fission, tunnel diode, Josephson junction, scanning tunneling microscope.

The transmission probability increases with decrease in height V_0 and width 'a' of the barrier. The transmission co-efficient is given as

$$T = 16\frac{E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-\alpha a}$$



<u>Case-2(E>V_0)</u>

According to quantum mechanics if E>V_0, all particles should be transmitted to the region-3 without any reflection but it is not possible for all the values of incident energy.

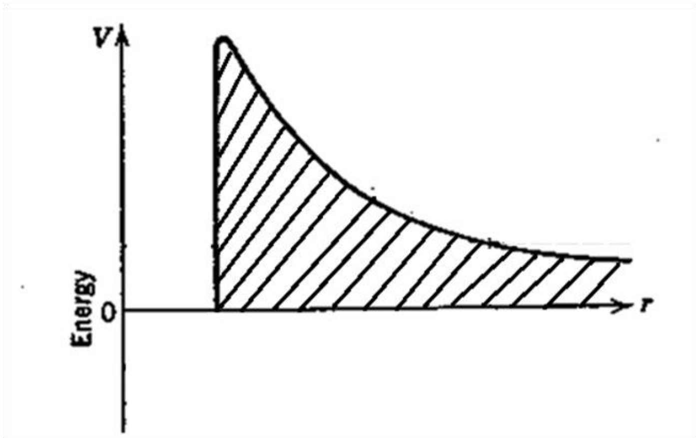
The transmission coefficient is one, for these values for which

$\frac{a}{\hbar}\sqrt{2m\left(E-\mathrm{V}_0\right)}=n\pi$, n=1,2,3,......

As $\lambda=\frac{h}{p}==\frac{h}{\sqrt{2m\left(E-V_0\right)}}$ in the region-2.

So, $\frac{2\pi a}{\lambda}=n\pi$

$\Rightarrow \boxed{a=n\frac{\lambda}{2}}$



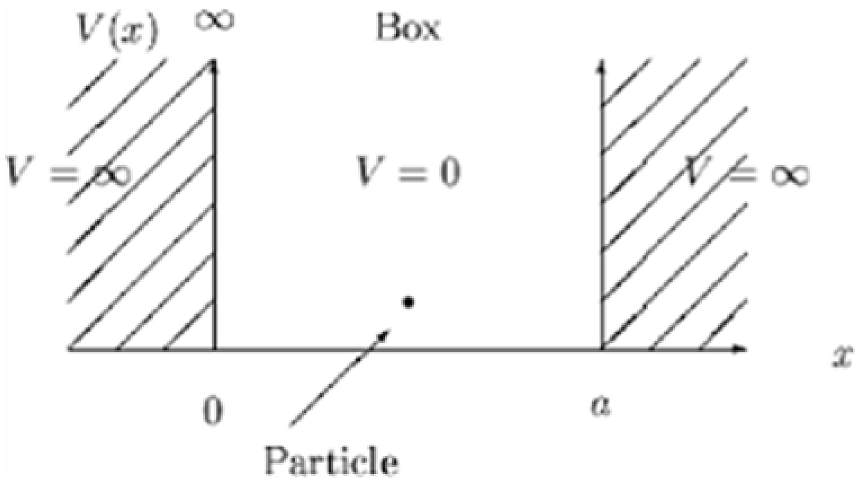
Particle in a one dimensional box:

The physical situation in which the potential between the boundary wall is zero and is infinite at the rigid walls is called one dimensional box or one dimensional infinite potential well.

The potential function for the situation is given as

$$V\left(x\right)=0, 0\le x<a$$

$$=\infty, x<0 \text{ and } x>a$$



Now Schrodinger equation inside the well is given as

$$\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}E\psi=0$$

$$\Rightarrow \frac{d^2\psi}{dx^2}+k^2\psi=0 \qquad (1)$$

The general solution of $eq^n(1)$ is given as

$$\psi(x)=C_1e^{ikx}+C_2e^{-ikx}$$

$$=A\sin kx+B\cos kx \qquad (2)$$

Where A and B are to be determined from the boundary condition at x=0 and x=a.

Thus $eq^n(2)$ becomes, $0=[A\sin kx+B\cos kx]_{x=0}=0+B$

$$\Rightarrow \boxed{B=0}$$

Thus the wave function inside the well is given as

$$\boxed{\psi(x)=A\sin kx} \qquad (0\le x\le a) \qquad (3)$$

<u>Energy eigen Values:-</u>

From $eq^n(3)$,

$$0=[A\sin kx]_{x=0}=A\sin ka \quad \text{at x=a}$$

$$\Rightarrow \boxed{ka = n\pi} \qquad \text{n=1,2,3....}$$

$$\Rightarrow \boxed{a = n\frac{\lambda}{2}}$$

Thus allowed bound states are possible for those energies for which the width of the potential well is equal to integral multiple of half wave length.

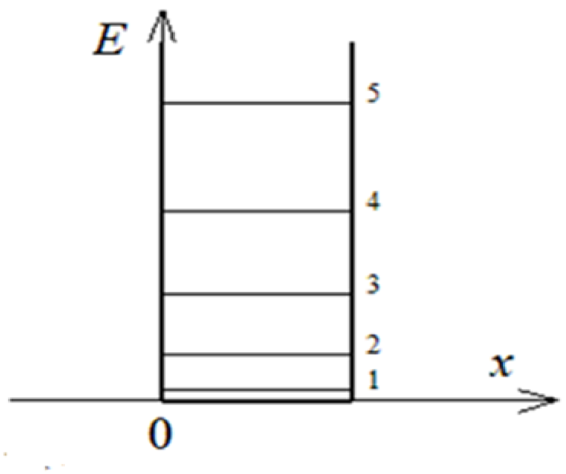
Since $k^2 = \frac{2mE}{\hbar^2}$

$$\Rightarrow k^2a^2 = \frac{2mEa^2}{\hbar^2} = n^2\pi^2$$

$$\Rightarrow \boxed{E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2}$$

Thus the energy of the particle in the infinite well is quantized.

- The ground state energy is $E_1 = \frac{\hbar^2\pi^2}{2ma^2}$ which is the minimum energy of the particle and is called the zero point energy.
- The energy of the higher allowed levels are multiple of E_1 and proportional to square of natural numbers.
- The energy levels are not equispaced.



<u>Eigen Functions</u>

The eigen functions of the allowed states can be obtained as

$$\int_{-\infty}^{+\infty} |\psi|^2 \, dx = 1$$

$$\Rightarrow \int_0^a |A|^2 \sin^2 kx dx = 1 \qquad (0 \le x \le a)$$

$$\Rightarrow |A|^2 \int_0^a \frac{1-\cos 2kx}{2} dx = 1$$

$$\Rightarrow \frac{2}{A^2} = [x]_0^a - \left[\frac{\sin 2kx}{2k}\right]_0^a = a$$

$$\Rightarrow A^2 = \frac{2}{a}$$

$$\Rightarrow A = \sqrt{\frac{2}{a}}$$

$$\therefore \boxed{\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} x}$$

Thus the eigen function for each quantum state are obtained by

$$\psi_1(x) = \sqrt{\frac{2}{a}} \sin \frac{\pi}{a} x$$

$$\psi_2(x) = \sqrt{\frac{2}{a}} \sin \frac{2\pi}{a} x$$

$$\psi_3(x) = \sqrt{\frac{2}{a}} \sin \frac{3\pi}{a} x \text{ etc.}$$

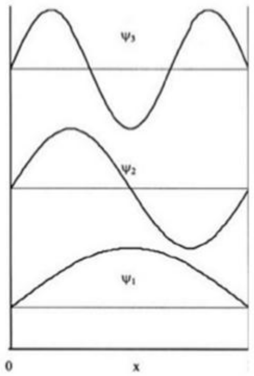


Fig. 3.1. The wave functions for the 0ne-dimensional particle-in-a-box

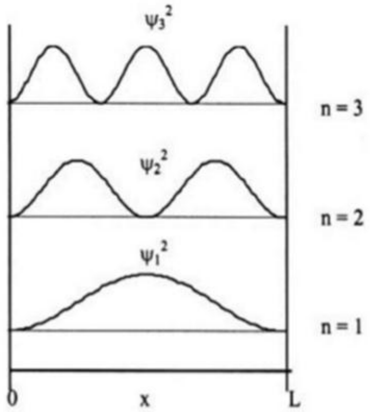


Fig. 3.2. The probability densities in 0ne-dimensional particle-in-a-box