VEER SURENDRA SAI UNIVERSITY OF TECHNOLOGY BURLA, ODISHA, INDIA

DEPARTMENT OF ELECTRICAL ENGINEERING

CONTROL SYSTEM ENGINEERING-II (3-1-0)

Lecture Notes

Subject Code: CSE-II

For 6th sem. Electrical Engineering & 7th Sem. EEE Student

DISCLAIMER

Department of Electrical Engineering,

CONTROL SYSTEM ENGINEERING-II (3-1-0)

MODULE-I (10 HOURS)

State Variable Analysis and Design: Introduction, Concepts of State, Sate Variables and State Model, State Models for Linear Continuous-Time Systems, State Variables and Linear Discrete-Time Systems, Diagonalization, Solution of State Equations, Concepts of Controllability and Observability, Pole Placement by State Feedback, Observer based state feedback control.

MODULE-II (10 HOURS)

Introduction of Design: The Design Problem, Preliminary Considerations of Classical Design, Realization of Basic Compensators, Cascade Compensation in Time Domain(Reshaping the Root Locus), Cascade Compensation in Frequency Domain(Reshaping the Bode Plot),
Introduction to Feedback Compensation and Robust Control System Design.
Digital Control Systems: Advantages and disadvantages of Digital Control, Representation of Sampled process, The z-transform, The z-transfer Function. Transfer function Models and dynamic response of Sampled-data closed loop Control Systems, The Z and S domain Relationship, Stability Analysis.

MODULE-III (10 HOURS)

Nonlinear Systems: Introduction, Common Physical Non-linearities, The Phase-plane Method: Basic Concepts, Singular Points, Stability of Nonlinear System, Construction of Phase-trajectories, The Describing Function Method: Basic Concepts, Derivation of Describing Functions, Stability analysis by Describing Function Method, Jump Resonance, Signal Stabilization.
Liapunov's Stability Analysis: Introduction, Liapunov's Stability Criterion, The Direct Method of Liapunov and the Linear System, Methods of Constructing Liapunov Functions for Nonlinear Systems, Popov's Criterion.

MODULE-IV (10 HOURS)

Optimal Control Systems: Introduction, Parameter Optimization: Servomechanisms, Optimal Control Problems: State Variable Approach, The State Regulator Problem, The Infinite-time Regulator Problem, The Output regulator and the Tracking Problems, Parameter Optimization: Regulators, Introduction to Adaptive Control.

BOOKS

[1]. K. Ogata, *"Modem Control Engineering",* PHI.
[2]. I.J. Nagrath, M. Gopal, *"Control Systems Engineering",* New Age International Publishers.
[3]. J.J.Distefano, III, A.R.Stubberud, I.J.Williams, *"Feedback and Control Systems",* TMH.
[4]. K.Ogata, *"Discrete Time Control System",* Pearson Education Asia.

MODULE-I

State space analysis.

State space analysis is an excellent method for the design and analysis of control systems. The conventional and old method for the design and analysis of control systems is the transfer function method. The transfer function method for design and analysis had many drawbacks.

Advantages of state variable analysis.

- It can be applied to non linear system.
- It can be applied to tile invariant systems.
- It can be applied to multiple input multiple output systems.
- Its gives idea about the internal state of the system.

State Variable Analysis and Design

State: The state of a dynamic system is the smallest set of variables called state variables such that the knowledge of these variables at time $t=t_o$ (Initial condition), together with the knowledge of input for $\geq t_0$, completely determines the behaviour of the system for any time $t \geq t_0$.

State vector: If n state variables are needed to completely describe the behaviour of a given system, then these n state variables can be considered the n components of a vector X. Such a vector is called a state vector.

State space: The n-dimensional space whose co-ordinate axes consists of the x_1 axis, x_2 axis,.... x_n axis, where x_1 , x_2 ,..... x_n are state variables: is called a state space.

State Model

Lets consider a multi input & multi output system is having

r inputs $u_1(t), u_2(t), \ldots \ldots . u_r(t)$

m no of outputs $y_1(t), y_2(t), \ldots \ldots . y_m(t)$

n no of state variables $x_1(t), x_2(t), \ldots \ldots . x_n(t)$

Then the state model is given by state & output equation

$$\dot{X}(t) = AX(t) + BU(t) \ldots \ldots \ldots \ldots \text{state equation}$$

$$Y(t) = CX(t) + DU(t) \ldots \ldots \ldots \text{output equation}$$

A is state matrix of size (n×n)

B is the input matrix of size (n×r)

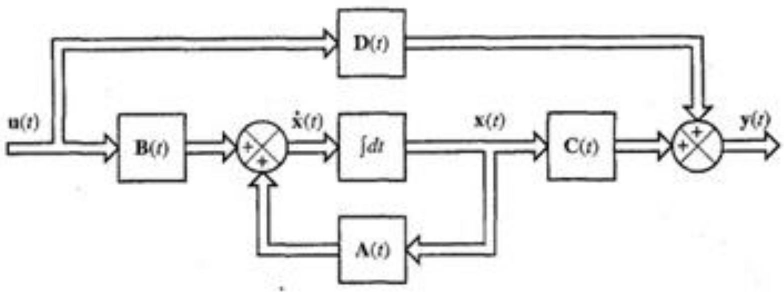
C is the output matrix of size (m×n)

D is the direct transmission matrix of size (m×r)

X(t) is the state vector of size (n×1)

Y(t) is the output vector of size (m×1)

U(t) is the input vector of size (r×1)



(Block diagram of the linear, continuous time control system represented in state space)

$$\dot{\mathbf{X}}(\mathbf{t}) = \mathbf{AX}(\mathbf{t}) + \mathbf{Bu}(\mathbf{t})$$

$$\mathbf{Y}(\mathbf{t}) = \mathbf{CX}(\mathbf{t}) + \mathbf{Du}(\mathbf{t})$$

STATE SPACE REPRESENTATION OF NTH ORDER SYSTEMS OF LINEAR DIFFERENTIAL EQUATION IN WHICH FORCING FUNCTION DOES NOT INVOLVE DERIVATIVE TERM

Consider following nth order LTI system relating the output y(t) to the input u(t).

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_{n-1} y^1 + a_n y = u$$

Phase variables: The phase variables are defined as those particular state variables which are obtained from one of the system variables & its (n-1) derivatives. Often the variables used is the system output & the remaining state variables are then derivatives of the output.

Let us define the state variables as

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \frac{dx}{dt}$$

$$x_3 = \frac{d\dot{y}}{dt} = \frac{dx_2}{dt}$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_n = y^{n-1} = \frac{dx_{n-1}}{dt}$$

From the above equations we can write

$$\dot{x_1} = x_2$$

$$\dot{x_2} = x_3$$

$$\vdots \qquad \qquad \vdots$$

$$\dot{x_{n-1}} = x_n$$

$$\dot{x_n} = -a_n x_1 - a_{n-1} x_2 - \cdots \ldots \ldots \ldots - a_1 x_n + u$$

Writing the above state equation in vector matrix form

$$\dot{\mathrm{X}}(\mathrm{t}) = \mathrm{AX}(\mathrm{t}) + \mathrm{Bu}(\mathrm{t})$$

Where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$, $A = \begin{bmatrix} 0 & 1 & 0 \ldots \ldots & 0 \\ 0 & 0 & 1 \ldots \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} \ldots \ldots . & -a_1 \end{bmatrix}_{n\times n}$

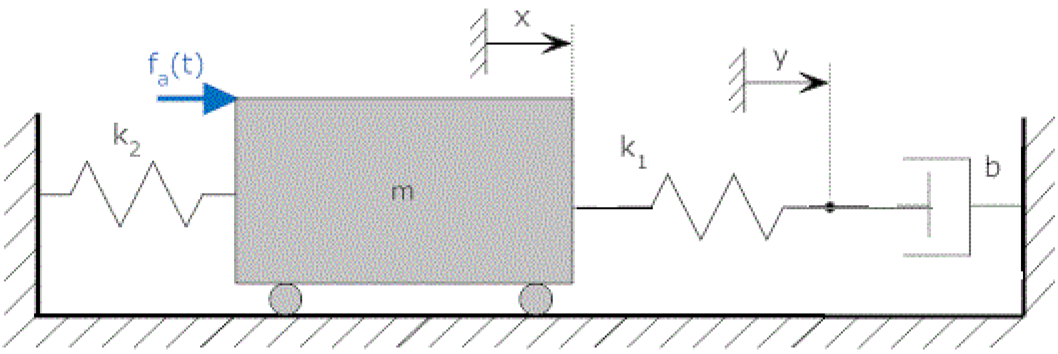
Output equation can be written as

$$\mathrm{Y}(\mathrm{t}) = \mathrm{CX}(\mathrm{t})$$

$$C = [1 \quad 0 \ldots \ldots . \quad 0]_{1\times n}$$

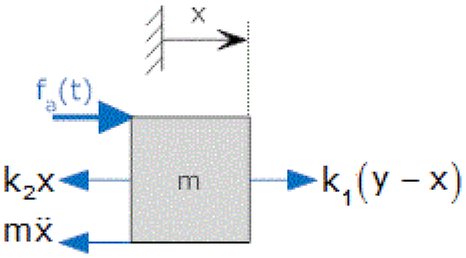
Example: Direct Derivation of State Space Model (Mechanical Translating)

Derive a state space model for the system shown. The input is f_a and the output is y.

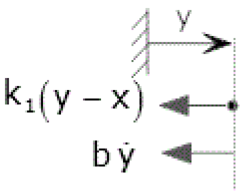


We can write free body equations for the system at x and at y.

Freebody Diagram **Equation**



$$m \cdot \ddot{x} + k_1 \cdot x + k_2 x - k_1 \cdot y = f_a$$



$$b \cdot \dot{y} + k_1 y - k_1 \cdot x = 0$$

There are three energy storage elements, so we expect three state equations. The energy storage elements are the spring, k_2, the mass, m, and the spring, k_1. Therefore we choose as our state variables x (the energy in spring k_2 is $\frac{1}{2}k_2x^2$), the velocity at x (the energy in the mass m is $\frac{1}{2}mv^2$, where v is the first derivative of x), and y (the energy in spring k_1 is $\frac{1}{2}k_1(y-x)^2$, so we could pick y-x as a state variable, but we'll just use y (since x is already a state variable; recall that the choice of state variables is not unique). Our state variables become:

$$\begin{aligned} q_1 &= x \\ q_2 &= \dot{x} \\ q_3 &= y \end{aligned}$$

Now we want equations for their derivatives. The equations of motion from the free body diagrams yield

$$\begin{aligned} \dot{q}_1 &= \dot{x} = q_2 \\ \dot{q}_2 &= \ddot{x} = \frac{1}{m}\left(f_a - k_1 x - k_2 x + k_1 y\right) \\ &= \frac{1}{m}\left(f_a - k_1 q_2 - k_2 q_1 + k_1 q_3\right) \\ \dot{q}_3 &= \dot{y} = \frac{k_1}{b}(x - y) = \frac{k_1}{b}(q_1 - q_3) \end{aligned}$$

or

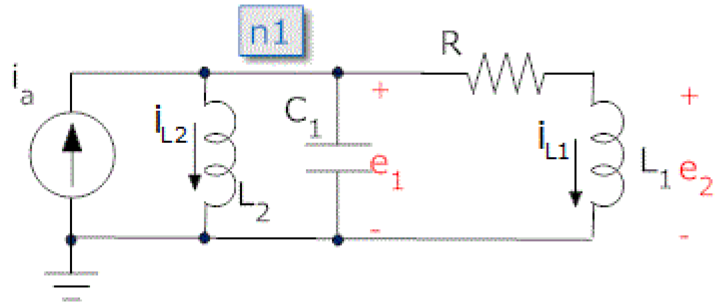
$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}u \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k_1 + k_2}{m} & 0 & \frac{k_1}{m} \\ \frac{k_1}{b} & 0 & -\frac{k_1}{b} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix}$$

$$y = \mathbf{C}\mathbf{q} + Du \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad D = 0$$

with the input $u=f_a$.

Example: Direct Derivation of State Space Model (Electrical)

Derive a state space model for the system shown. The input is i_a and the output is e_2.



There are three energy storage elements, so we expect three state equations. Try choosing i_1, i_2 and e_1 as state variables. Now we want equations for their derivatives. The voltage across the inductor L_2 is e_1 (which is one of our state variables)

$$L_2 \frac{di_{L2}}{dt} = e_1$$

so our first state variable equation is

$$\frac{di_{L2}}{dt} = \frac{1}{L_2} e_1$$

If we sum currents into the node labeled n1 we get

$$i_a - i_{L2} - i_{C1} - i_{L1} = 0$$

This equation has our input (ia) and two state variable (iL2 and iL1) and the current through the capacitor. So from this we can get our second state equation

$$i_{C1} = C_1 \frac{de_1}{dt} = i_a - i_{L2} - i_{L1}$$

$$\frac{de_1}{dt} = \frac{1}{C_1}\left(i_a - i_{L2} - i_{L1}\right)$$

Our third, and final, state equation we get by writing an equation for the voltage across L_1 (which is e_2) in terms of our other state variables

$$e_2 = L_1 \frac{di_{L1}}{dt} = e_1 - Ri_{L1}$$

$$\frac{di_{L1}}{dt} = \frac{1}{L_1}\left(e_1 - Ri_{L1}\right)$$

We also need an output equation:

$$e_2 = e_1 - Ri_{L1}$$

So our state space representation becomes

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} i_{L2} \\ e_2 \\ i_{L1} \end{bmatrix}$$

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}u \qquad \mathbf{A} = \begin{bmatrix} 0 & \frac{1}{L_2} & 0 \\ -\frac{1}{C_1} & 0 & -\frac{1}{C_1} \\ 0 & \frac{1}{L_1} & -\frac{R}{L_1} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{C_1} \\ 0 \end{bmatrix}$$

$$y = \mathbf{C}\mathbf{q} + Du \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 & -R \end{bmatrix} \qquad D = 0$$

State Space to Transfer Function

Consider the state space system:

$$\begin{aligned} \dot{\mathbf{q}}(t) &= \mathbf{A}\mathbf{q}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{q}(t) + Du(t) \end{aligned}$$

Now, take the Laplace Transform (with zero initial conditions since we are finding a transfer function):

$$\begin{aligned} s\mathbf{Q}(s) &= \mathbf{A}\mathbf{Q}(s) + \mathbf{B}U(s) \\ Y(s) &= \mathbf{C}\mathbf{Q}(s) + DU(s) \end{aligned}$$

We want to solve for the ratio of Y(s) to U(s), so we need so remove **Q**(s) from the output equation. We start by solving the state equation for **Q**(s)

$$s\mathbf{Q}(s) - A\mathbf{Q}(s) = \mathbf{B}U(s)$$
$$(s\mathbf{I} - \mathbf{A})\mathbf{Q}(s) = \mathbf{B}U(s)$$
$$\mathbf{Q}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) = \mathbf{\Phi}(s)\mathbf{B}U(s); \quad \text{where } \mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$$

The matrix **Φ**(s) is called the state transition matrix. Now we put this into the output equation

$$\begin{aligned} Y(s) &= \mathbf{C\Phi}(s)\mathbf{B}U(s) + DU(s) \\ &= (\mathbf{C\Phi}(s)\mathbf{B} + D)U(s) \end{aligned}$$

Now we can solve for the transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C\Phi}(s)\mathbf{B} + D = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

Note that although there are many state space representations of a given system, all of those representations will result in the same transfer function (i.e., the transfer function of a system is unique; the state space representation is not).

Example: State Space to Transfer Function

Find the transfer function of the system with state space representation

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -2 \end{bmatrix}\mathbf{q} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \mathbf{C}\mathbf{q} + Du = \begin{bmatrix} 5 & 1 & 0 \end{bmatrix} + 0 \cdot u$$

First find (s**I**-**A**) and the $\mathbf{\Phi}=(s\mathbf{I}-\mathbf{A})^{-1}$ (note: this calculation is not obvious. Details are here). Rules for inverting a 3x3 matrix are here.

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 3 & 4 & s+2 \end{bmatrix}$$

$$\mathbf{\Phi} = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{\begin{bmatrix} s^2+2s+4 & 2+s & 1 \\ -3 & s(2+s) & s \\ -3s & -3-4s & s^2 \end{bmatrix}}{s^3+2s^2+4s+3}$$

Now we can find the transfer function

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C\Phi B} + D$$
$$= \frac{s+5}{s^3+2s^2+4s+3}$$

To make this task easier, MatLab has a command (**ss2tf**) for converting from state space to transfer function.

```
>> % First define state space system

>> A=[0 1 0; 0 0 1; -3 -4 -2];

>> B=[0; 0; 1];

>> C=[5 1 0];

>> [n,d]=ss2tf(A,B,C,D)



n =
```

```
            0            0     1.0000
5.0000

d =

     1.0000     2.0000     4.0000
3.0000



>> mySys_tf=tf(n,d)

Transfer function:

       s + 5

----------------------

 s^3 + 2 s^2 + 4 s + 3
```

Transfer Function to State Space

Recall that state space models of systems are not unique; a system has many state space representations. Therefore we will develop a few methods for creating state space models of systems.

Before we look at procedures for converting from a transfer function to a state space model of a system, let's first examine going from a differential equation to state space. We'll do this first with a simple system, then move to a more complex system that will demonstrate the usefulness of a standard technique.

First we start with an example demonstrating a simple way of converting from a single differential equation to state space, followed by a conversion from transfer function to state space.

Example: Differential Equation to State Space (simple)

Consider the differential equation with no derivatives on the right hand side. We'll use a third order equation, thought it generalizes to n^{th} order in the obvious way.

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0 u$$

For such systems (no derivatives of the input) we can choose as our n state variables the variable y and its first n-1 derivatives (in this case the first two derivatives)

$$q_1 = y$$
$$q_2 = \dot{y}$$
$$q_3 = \ddot{y}$$

Taking the derivatives we can develop our state space model

$$\begin{aligned}\dot{q}_1 &= q_2 = \dot{y}\\ \dot{q}_2 &= q_3 = \ddot{y}\\ \dot{q}_3 &= \dddot{y} = -a_3 y - a_2\dot{y} - a_1\ddot{y} + b_0 u\\ &= -a_3 q_1 - a_2 q_2 - a_1 q_3 + b_0 u\end{aligned}$$

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\mathbf{q} + \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix}u$$

$$y = \mathbf{Cq} + Du = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{q} + 0 \cdot u$$

*Note: For an nth order system the matrices generalize in the obvious way (**A** has ones above the main diagonal and the differential equation constants for the last row, **B** is all zeros with b_0 in the bottom row, **C** is zero except for the leftmost element which is one, and D is zero)*

Repeat Starting from Transfer Function

Consider the transfer function with a constant numerator (note: this is the same system as in the preceding example). We'll use a third order equation, thought it generalizes to n^{th} order in the obvious way.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\left(s^3 + a_1 s^2 + a_2 s + a_3\right)Y(s) = b_0 U(s)$$

For such systems (no derivatives of the input) we can choose as our n state variables the variable y and its first n-1 derivatives (in this case the first two derivatives)

$$\begin{aligned} q_1(t) &= y(t) \qquad & Q_1(s) &= Y(s)\\ q_2(t) &= \dot{y}(t) \qquad & Q_2(s) &= sY(s)\\ q_3(t) &= \ddot{y}(t) \qquad & Q_3(s) &= s^2Y(s)\end{aligned}$$

Taking the derivatives we can develop our state space model (which is exactly the same as when we started from the differential equation).

$$
\begin{aligned}
sQ_1(s) &= Q_2(s) = sY(s) \\
sQ_2(s) &= Q_3(s) = s^2Y(s) \\
sQ_3(s) &= s^3Y(s) = -a_1s^2Y(s) - a_2sY(s) - a_3Y(s) + b_0u \\
&= -a_1s^2Q_3(s) - a_2sQ_2(s) - a_3Q_1(s) + b_0u
\end{aligned}
$$

$$
s\mathbf{Q}(s) = \mathbf{A}\mathbf{Q}(s) + \mathbf{B}U(s) == \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\mathbf{Q}(s) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s)
$$

$$
Y(s) = \mathbf{C}\mathbf{Q}(s) + DU(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{Q}(s) + 0 \cdot U(s)
$$

*Note: For an nth order system the matrices generalize in the obvious way (**A** has ones above the main diagonal and the coefficients of the denominator polynomial for the last row, **B** is all zeros with b_0 (the numerator coefficient) in the bottom row, **C** is zero except for the leftmost element which is one, and D is zero)*

If we try this method on a slightly more complicated system, we find that it initially fails (though we can succeed with a little cleverness).

Example: Differential Equation to State Space (harder)

Consider the differential equation with a single derivative on the right hand side.

$$
\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3y = b_0\dot{u} + b_1u
$$

We can try the same method as before:

$$
\begin{aligned}
q_1 &= y \\
q_2 &= \dot{y} \\
q_3 &= \ddot{y}
\end{aligned}
$$

$$
\begin{aligned}
\dot{q}_1 &= q_2 = \dot{y} \\
\dot{q}_2 &= q_3 = \ddot{y} \\
\dot{q}_3 &= \dddot{y} = -a_3y - a_2\dot{y} - a_1\ddot{y} + b_0\dot{u} + b_1u \\
&= -a_3q_1 - a_2q_2 - a_1q_3 + b_0\dot{u} + b_1u
\end{aligned}
$$

The method has failed because there is a derivative of the input on the right hand, and that is not allowed in a state space model.

Fortunately we can solve our problem by revising our choice of state variables.

$$
\begin{aligned}
q_1 &= y \\
q_2 &= \dot{y} \\
q_3 &= \ddot{y} - b_0u
\end{aligned}
$$

Now when we take the derivatives we get:

$$\begin{aligned}\dot{q}_1 &= q_2 = \dot{y} \\ \dot{q}_2 &= \ddot{y} \\ \dot{q}_3 &= \dddot{y} - b_0\dot{u} = -a_3 y - a_2\dot{y} - a_1\ddot{y} + b_1 u\end{aligned}$$

The second and third equations are not correct, because $\ddot{y}$ is not one of the state variables. However we can make use of the fact:

$$\begin{aligned}q_3 &= \dot{y} - b_0 u, \quad \text{so} \\ \ddot{y} &= q_3 + b_0 u\end{aligned}$$

The second state variable equation then becomes

$$\dot{q}_2 = q_3 + b_0 u$$

In the third state variable equation we have successfully removed the derivative of the input from the right side of the third equation, and we can get rid of the $\ddot{y}$ term using the same substitution we used for the second state variable.

$$\begin{aligned}\dot{q}_3 &= -a_3 y - a_2\dot{y} - a_1\ddot{y} + b_1 u \\ &= -a_3 q_1 - a_2 q_2 - a_1\left(q_3 + b_0 u\right) + b_1 u \\ &= -a_3 q_1 - a_2 q_2 - a_1 q_3 + \left(b_1 - a_1 b_0\right) u\end{aligned}$$

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\mathbf{q} + \begin{bmatrix} 0 \\ b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \mathbf{Cq} + Du = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{q} + 0 \cdot u$$

The process described in the previous example can be generalized to systems with higher order input derivatives but unfortunately gets increasingly difficult as the order of the derivative increases. When the order of derivatives is equal on both sides, the process becomes much more difficult (and the variable "D" is no longer equal to zero). Clearly more straightforward techniques are necessary. Two are outlined below, one generates a state space method known as the "controllable canonical form" and the other generates the "observable canonical form (the meaning of these terms derives from Control Theory but are not important to us).

Controllable Canonical Form (CCF)

Probably the most straightforward method for converting from the transfer function of a system to a state space model is to generate a model in "controllable canonical form." This term comes from Control Theory but its exact meaning is not important to us. To see how this method of generating a state space model works, consider the third order differential transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^3 + a_1 s^2 + a_2 s + a_3}$$

We start by multiplying by Z(s)/Z(s) and then solving for Y(s) and U(s) in terms of Z(s). We also convert back to a differential equation.

$$Y(s) = \left(b_0 s^2 + b_1 s + b_2\right) Z(s) \qquad y = b_0 \ddot{z} + b_1 \dot{z} + b_2 z$$

$$U(s) = \left(s^3 + a_1 s^2 + a_2 s + a_3\right) Z(s) \qquad u = \dddot{z} + a_1 \ddot{z} + a_2 \dot{z} + a_3 z$$

We can now choose z and its first two derivatives as our state variables

$$\begin{aligned} q_1 &= z & \dot{q}_1 &= \dot{z} = q_2 \\ q_2 &= \dot{z} & \dot{q}_2 &= \ddot{z} = q_3 \\ q_3 &= \ddot{z} & \dot{q}_3 &= \dddot{z} = u - a_1 \ddot{z} - a_2 \dot{z} - a_3 z \\ & & &= u - a_1 q_3 - a_2 q_2 - a_3 q_1 \end{aligned}$$

Now we just need to form the output

$$\begin{aligned} y &= b_0 \ddot{z} + b_1 \dot{z} + b_2 z \\ &= b_0 q_3 + b_1 q_2 + b_2 q_1 \end{aligned}$$

From these results we can easily form the state space model:

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{q} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \mathbf{Cq} + Du = \begin{bmatrix} b_2 & b_1 & b_0 \end{bmatrix} \mathbf{q} + 0 \cdot u$$

In this case, the order of the numerator of the transfer function was less than that of the denominator. If they are equal, the process is somewhat more complex. A result that works in all cases is given below; the details are here. For a general n^{th} order transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

the controllable canonical state space model form is

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{B}u; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y = \mathbf{Cq} + Du \qquad \mathbf{C} = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdots & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \quad D = b_0$$

Key Concept: Transfer function to State Space (CCF)

For a general n^{th} order transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

the controllable canonical state space model form is

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}u; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y = \mathbf{C}\mathbf{q} + Du \quad \mathbf{C} = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdots & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \quad D = b_0$$

Observable Canonical Form (OCF)

Another commonly used state variable form is the "observable canonical form." This term comes from Control Theory but its exact meaning is not important to us. To understand how this method works consider a third order system with transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^3 + a_1 s^2 + a_2 s + a_3}$$

We can convert this to a differential equation and solve for the highest order derivative of y:

$$\left(s^3 + a_1 s^2 + a_2 s + a_3\right) Y(s) = \left(b_0 s^2 + b_1 s + b_2\right) U(s)$$

$$s^3 Y(s) = \left(b_0 s^2 + b_1 s + b_2\right) U(s) - \left(a_1 s^2 + a_2 s + a_3\right) Y(s)$$

$$\dddot{y} = b_0 \ddot{u} + b_1 \dot{u} + b_2 u - a_1 \ddot{y} - a_2 \dot{y} - a_3 y$$

Now we integrate twice (the reason for this will be apparent soon), and collect terms according to order of the integral:

$$\dot{y} = b_0 u + b_1 \int u \cdot dt + b_2 \int\int u \cdot dt \cdot dt - a_1 y - a_2 \int y \cdot dt - a_3 \int\int y \cdot dt \cdot dt$$

$$= b_0 u - a_1 y + \int \left(b_1 u - a_2 y\right) dt + \int\int \left(b_2 u - a_3 y\right) \cdot dt \cdot dt$$

Choose the output as our first state variable

$$q_1 = y \qquad \dot{q}_1 = \dot{y} = b_0 u - a_1 y + \int \left(b_1 u - a_2 y\right) dt + \int\int \left(b_2 u - a_3 y\right) \cdot dt \cdot dt$$

Looking at the right hand side of the differential equation we note that $y=q_1$ and we call the two integral terms q_2:

$$q_2 = \int \left(b_1 u - a_2 y\right) dt + \int\int \left(b_2 u - a_3 y\right) dt \cdot dt$$

so

$$\dot{q}_1 = \dot{y} = b_0 u - a_1 q_1 + q_2$$

This is our first state variable equation.

Now let's examine q_2 and its derivative:

$$q_2 = \int (b_1 u - a_2 y)\,dt + \int\int (b_2 u - a_3 y)\,dt \cdot dt$$
$$\dot{q}_2 = b_1 u - a_2 y + \int (b_2 u - a_3 y)\,dt$$

Again we note that $y=q_1$ and we call the integral terms q_3:

$$q_3 = \int (b_2 u - a_3 y)\,dt$$

so

$$\dot{q}_2 = b_1 u - a_2 q_1 + q_3$$

This is our second state variable equation.

Now let's examine q_3 and its derivative:

$$\begin{aligned} q_3 &= \int (b_2 u - a_3 y)\,dt \\ \dot{q}_3 &= b_2 u - a_3 y \\ &= b_2 u - a_3 q_1 \end{aligned}$$

This is our third, and last, state variable equation.

Our state space model now becomes:

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}u = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \mathbf{q} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} u$$
$$y = \mathbf{C}\mathbf{q} + Du = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{q} + 0 \cdot u$$

In this case, the order of the numerator of the transfer function was less than that of the denominator. If they are equal, the process is somewhat more complex. A result that works in all cases is given below; the details are here. For a general n^{th} order transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

the observable canonical state space model form is

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}u; \quad \mathbf{A} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ -a_{n-1} & 0 & \vdots & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix}$$

$$y = \mathbf{C}\mathbf{q} + Du \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad D = b_0$$

Key Concept: Transfer function to State Space (OCF)

For a general n^{th} order transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

the observable canonical state space model form is

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{B}u; \quad \mathbf{A} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ -a_{n-1} & 0 & \vdots & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix}$$

$$y = \mathbf{Cq} + Du \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad D = b_0$$

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C\Phi}(s)\mathbf{B} + D = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$= \frac{\mathbf{C}[\mathrm{Adj}(SI - A)\mathbf{B}] + |SI - A|D}{|SI - A|}$$

$|SI - A|$ is also known as characteristic equation when equated to zero.

MATLab Code

Transfer Function to State Space(**tf2ss**)

$$\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160}$$

```
num=[1 0];
den=[1 14 56 160];
[A,B,C,D]=tf2ss(num,den)
```

A =

```
  -14   -56  -160
    1     0     0
    0     1     0
```

B =

1

0

0

C =

0 1 0

D =

0

Concept of Eigen Values and Eigen Vectors

The roots of characteristic equation that we have described above are known as eigen values of matrix A.
Now there are some properties related to eigen values and these properties are written below-

1. Any square matrix A and its transpose A^T have the same eigen values.
2. Sum of eigen values of any matrix A is equal to the trace of the matrix A.
3. Product of the eigen values of any matrix A is equal to the determinant of the matrix A.
4. If we multiply a scalar quantity to matrix A then the eigen values are also get multiplied by the same value of scalar.
5. If we inverse the given matrix A then its eigen values are also get inverses.
6. If all the elements of the matrix are real then the eigen values corresponding to that matrix are either real or exists in complex conjugate pair.

Eigen Vectors

Any non zero vector m_i that satisfies the matrix equation $(\lambda_i I - A)m_i = 0$ is called the eigen vector of A associated with the eigen value λ_i,. Whereλ_i, i = 1, 2, 3, ……..n denotes the i^{th} eigen values of A.

This eigen vector may be obtained by taking cofactors of matrix $(\lambda_i I - A)$ along any row & transposing that row of cofactors.

Diagonalization

Let $m_1, m_2, \dots \dots .. m_n$ be the eigenvectors corresponding to the eigen value $\lambda_1, \lambda_2, \dots \dots .. \lambda_n$ respectively.

Then $M = [m_1 \; \vdots \; m_2 \; \vdots \cdots \; \vdots \; m_n]$ is called diagonalizing or **modal matrix** of A.

Consider the n[th] order MIMO state model

$$\dot{X}(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t) + DU(t)$$

System matrix A is non diagonal, so let us define a new state vector V(t) such that X(t)=MV(t).

Under this assumption original state model modifies to

$$\dot{V}(t) = \tilde{A}V(t) + \tilde{B}U(t)$$

$$Y(t) = \check{C}V(t) + DU(t)$$

Where $\tilde{\boldsymbol{A}} = \boldsymbol{M^{-1}AM} = diagonal\ matrix,\quad \tilde{\boldsymbol{B}} = \boldsymbol{M^{-1}B},\quad \tilde{\boldsymbol{C}} = \boldsymbol{CM}$

The above transformed state model is in canonical state model. The transformation described above is called similarity transformation.

If the system matrix A is in companion form & if all its n eigen values are distinct, then modal matrix will be special matrix called the **Vander Monde matrix.**

$$Vander\ Monde\ matrix\ V = \begin{bmatrix} 1 & 1 & 1 \dots \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \dots \dots & \lambda_n \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \lambda_3^{n-2} \dots \dots & \lambda_n^{n-2} \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} \dots \dots . & \lambda_n^{n-1} \end{bmatrix}_{n \times n}$$

State Transition Matrix and Zero State Response

We are here interested in deriving the expressions for the state transition matrix and zero state response. Again taking the state equations that we have derived above and taking their Laplace transformation we have,

$$sX(s) - X(0) = AX(s) + BU(s)$$

Now on rewriting the above equation we have

$$X(s) = [sI - A]^{-1} \times x(0) + [sI - A]^{-1} \times BU(s)$$

Let $[sI-A]^{-1} = \theta(s)$ and taking the inverse Laplace of the above equation we have

$$X(t) = \theta(t).x(0) + L^{-1} \times \theta(s)BU(s)$$

The expression θ(t) is known as **state transition matrix(STM).**

$L^{-1}.\theta(t)BU(s)$ = zero state response.

Now let us discuss some of the properties of the state transition matrix.

1. If we substitute t = 0 in the above equation then we will get 1.Mathematically we can write θ(0) =1.
2. If we substitute t = -t in the θ(t) then we will get inverse of θ(t). Mathematically we can write $\theta(-t) = [\theta(t)]^{-1}$.
3. We also another important property $[\theta(t)]^n = \theta(nt)$.

Computation of STN using Cayley-Hamilton Theorem

The Cayley–hamilton theorem states that every square matrix A satisfies its own characteristic equation.
This theorem provides a simple procedure for evaluating the functions of a matrix.
To determine the matrix polynomial

$$f(A) = k_0 I + k_1 A + k_2 A^2 + \cdots + k_n A^n + k_{n+1} A^{n+1} + \cdots$$

Consider the scalar polynomial

$$f(\lambda) = k_0 + k_1\lambda + k_2\lambda^2 + \cdots + k_n\lambda^n + k_{n+1}\lambda^{n+1} + \cdots$$

Here A is a square matrix of size (n×n). Its characteristic equation is given by

$$q(\lambda) = |\lambda I - A| = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$$

If $f(A)$ is divided by the characteristic polynomial $q(\lambda)$, then

$$\frac{f(\lambda)}{q(\lambda)} = Q(\lambda) + \frac{R(\lambda)}{q(\lambda)}$$

$$f(\lambda) = Q(\lambda)q(\lambda) + R(\lambda) \quad \ldots\ldots\ldots\ldots\ldots..(1)$$

Where $R(\lambda)$ is the remainder polynomial of the form

$$R(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_{n-1}\lambda^{n-1} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

If we evaluate $f(A)$ at the eigen values $\lambda_1, \lambda_2, \dots\dots..\lambda_n$, then $q(\lambda) = 0$ and we have from equation (1) $f(\lambda_i) = R(\lambda_i); \quad i = 1,2, \dots.n \qquad \dots\dots\dots\dots\dots\dots..(3)$

The coefficients $a_0, a_1, \dots\dots..a_{n-1}$, can be obtained by successfully substituting $\lambda_1, \lambda_2, \dots\dots..\lambda_n$ into equation (3).
Substituting A for the variable λ in equation (1), we get

$$f(A) = Q(A)q(A) + R(A)$$

As $q(A)\ is\ zero, so\ f(A) = R(A)$

$$\Rightarrow \quad f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_{n-1} A^{n-1}$$

$$which\ is\ the\ desired\ result.$$

CONCEPTS OF CONTROLLABILITY & OBSERVABILITY

State Controllability

A system is said to be completely state controllable if it is possible to transfer the system state from any initial state $X(t_o)$ to any desired state X(t) in specified finite time by a control vector u(t).

Kalman's test

Consider n^{th} order multi input LTI system with m dimensional control vector

$$\dot{X}(t) = AX(t) + BU(t)$$

is completely controllable if & only if the rank of the composite matrix Q_c is n.

$$\boldsymbol{Q_c = [B \ \vdots \ AB \ \vdots \cdots \vdots \ A^{n-1}B]}$$

Observability

A system is said to be completely observable, if every state $X(t_o)$ can be completely identified by measurements of the outputs y(t) over a finite time interval$(t_o \le t \le t_1)$.

Kalman's test

Consider n^{th} order multi input LTI system with m dimensional output vector

$$\dot{X}(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t) + DU(t)$$

is completely observable if & only if the rank of the observability matrix Q_o is n.

$$\boldsymbol{Q_o = [C^T \ \vdots \ A^T C^T \ \vdots \cdots \vdots \ (A^T)^{n-1} C^T]}$$

Principle of Duality: It gives relationship between controllability & observability.

- The Pair (AB) is controllable implies that the pair ($A^T B^T$) is observable.
- The pair (AC) is observable implies that the pair ($A^T C^T$) is controllable.

Design of Control System in State Space

Pole placement at State Space

Assumptions:

- The system is completely state controllable. f
- The state variables are measurable and are available for feedback. f
- Control input is unconstrained.

Objective:

The closed loop poles should lie at $\mu_1, \mu_2, \dots \dots . . \mu_n$ which are their 'desired locations'

Necessary and sufficient condition: The system is completely state controllable.

Consider the system

$$\dot{X}(t) = AX(t) + BU(t)$$

The control vector U is designed in the following state feedback form U =-KX

This leads to the following closed loop system

$$\dot{X}(t) = (A - BK)X(t) = A_{CL}X(t)$$

The gain matrix K is designed in such a way that

$$|SI - (A - BK)| = (S - \mu_1)(S - \mu_2) \dots . . (S - \mu_n)$$

Pole Placement Design Steps:Method 1 (low order systems, n ≤ 3):

- Check controllability
- Define $K = [k_1 \quad k_2 \quad k_3]$
- Substitute this gain in the desired characteristic polynomial equation

$$|SI - A + BK| = (S - \mu_1)(S - \mu_2) \dots . . (S - \mu_n)$$

- Solve for k_1, k_2, k_3 by equating the like powers of S on both sides

MATLab Code

Finding State Feedback gain matrix with MATLab

MATLab code **acker** is based on Ackermann's formula and works for single input single output system only.

MATLab code **place** works for single- or multi-input system.

Example

Consider the system with state equation

$$\dot{X}(t) = AX(t) + BU(t)$$

Where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

By using state feedback control u=-Kx, it is desired to have the closed –loop poles at

$$\mu_1 = -2 + j4, \qquad \mu_2 = -2 - j4, \qquad \mu_3 = -10$$

Determine the state feedback gain matrix K with MATLab

```
A=[0 1 0;0 0 1;-1 -5 -6];
B=[0;0;1];
J=[-2+i*4 -2-i*4 -10];
k=acker(A,B,J)
```

```
k =

  199    55     8
```

```
A=[0 1 0;0 0 1;-1 -5 -6];
B=[0;0;1];
J=[-2+i*4 -2-i*4 -10];
k=place(A,B,J)
```

```
k =


  199.0000   55.0000    8.0000
```

State Estimators or Observers

• One should note that although state feed back control is very attractive because of precise computation of the gain matrix *K*, implementation of a state feedback controller is possible only when all state variables are directly measurable with help of some kind of sensors.

• Due to the excess number of required sensors or unavailability of states for measurement, in most of the practical situations this requirement is not met.

• Only a subset of state variables or their combinations may be available for measurements. Sometimes only output *y* is available for measurement.

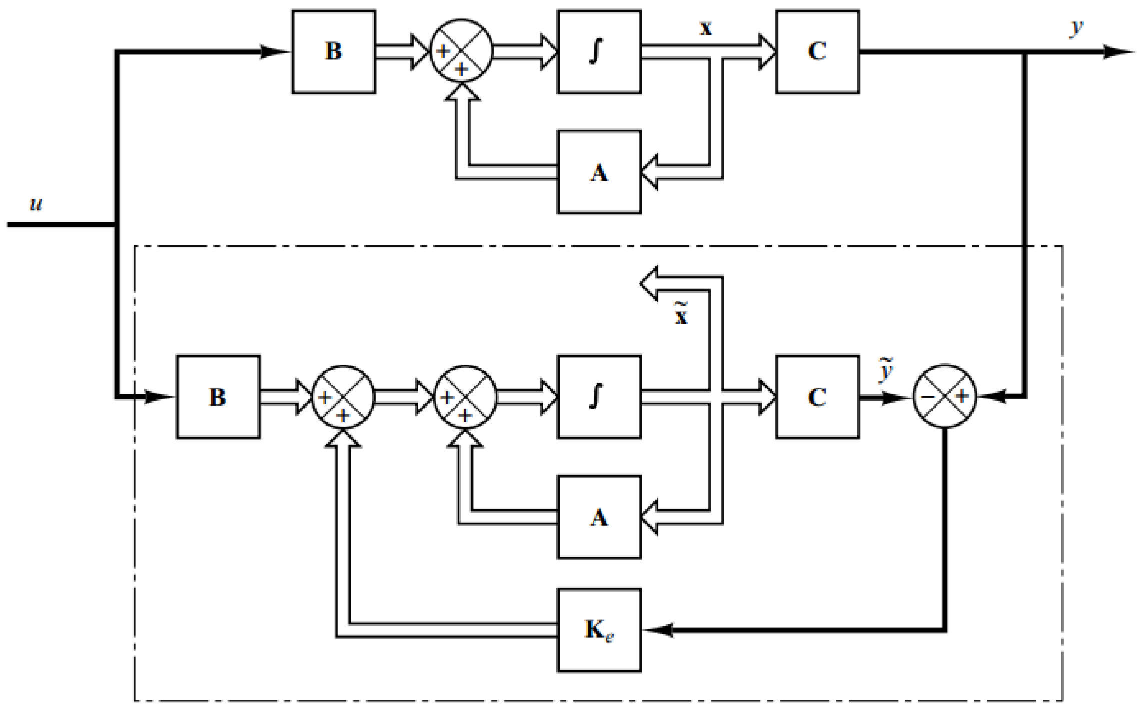
• Hence the need for an estimator or observer is obvious which estimates all state variables while observing input and output.

Full Order Observer : If the state observer estimates all the state variables, regardless of whether some are available for direct measurements or not, it is called a full order observer.

Reduced Order Observer : An observer that estimates fewer than ``n" states of the system is called reduced order observer.

Minimum Order Observer : If the order of the observer is minimum possible then it is called minimum order observer.

Observer Block Diagram



Full-order state observer

Design of an Observer

The governing equation for a dynamic system (Plant) in statespace representation may be written as:

$$\dot{X}(t) = AX(t) + BU(t)$$
$$Y(t) = CX(t)$$

The governing equation for the Observer based on the block diagram is shown below. The superscript '~' refers to estimation.

$$\dot{\tilde{X}} = A\tilde{X} + BU + K_e(Y - \tilde{Y})$$
$$\tilde{Y} = C\tilde{X}$$

Define the error in estimation of state vector as

$$e = (X - \tilde{X})$$

The error dynamics could be derived now from the observer governing equation and state space equations for the system as:

$$\dot{e} = (A - K_e C)e$$

$$Y - \tilde{Y} = Ce$$

The corresponding characteristic equation may be written as:

$$|SI - (A - K_e C)\ | = 0$$

You need to design the observer gains such that the desired error dynamics is obtained.

Observer Design Steps:Method 1 (low order systems, n ≤ 3):

- Check the observability
- Define $K_e = \begin{bmatrix} k_{e1} \\ k_{e2} \\ k_{e3} \end{bmatrix}$
- Substitute this gain in the desired characteristic polynomial equation

$$|SI - (A - K_e C)\ | = (S - \mu_1)(S - \mu_2) \ldots\ldots (S - \mu_n)$$

- Solve for k_1, k_2, k_3 by equating the like powers of S on both sides

Here $\mu_1, \mu_2, \ldots\ldots\ldots \mu_n$ are desired eigen values of observer matrix.

MODEL QUESTIONS

Module-1

Short Questions each carrying Two marks.

1. The System matrix of a continuous time system, described in the state variable form is

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

Determine the range of x & y so that the system is stable.

2. For a single input system

$$\dot{X} = AX + BU$$
$$Y = CX$$
$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Check the controllability & observability of the system.

3. Given the homogeneous state space equation $\dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} X$;

Determine the steady state value $X_{ss} = \lim_{t\to\infty} X(t)$ given the initial state value

$$X(0) = \begin{bmatrix} 10 \\ -10 \end{bmatrix}.$$

4. State Kalman's test for observability.

The figures in the right-hand margin indicate marks.

5. For a system represented by the state equation $\dot{X} = AX$

Where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$$

Find the initial condition state vector X(0) which will excite only the mode corresponding to the eigenvalue with the most negative real part. **[10]**

6. Write short notes on Properties of state transition matrix. **[3.5]**

7. Investigate the controllability and observability of the following system:

$$\dot{X} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u;\ Y = \begin{bmatrix} 0 & 1 \end{bmatrix} X$$ **[8]**

8. Write short notes on **[4×5]**

(a) Pole placement by state feedback.
(b) state transition matrix
(c) MIMO systems
(d) hydraulic servomotor
(e) Principle of duality due to kalman

9. A system is described by the following differential equation. Represent the system in state space:

$$\frac{d^3X}{dt^3}+3\frac{d^2X}{dt^2}+4\frac{dX}{dt}+4X=u_1(t)+3u_2(t)+4u_3(t)$$

and outputs are

$$y_1=4\frac{dX}{dt}+3u_1$$

$y_2=\frac{d^2X}{dt^2}+4u_2+u_3$ **[8]**

10. Find the time response of the system described by the equation

$\dot{X}(t)=\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}X+\begin{bmatrix}0\\1\end{bmatrix}u(t)$

$X(0)=\begin{bmatrix}-1\\0\end{bmatrix}, u(t)=1, t>0$ **[14]**

11. (a) Obtain a state space representation of the system

$\frac{C(s)}{U(s)}=\frac{10(s+2)}{s^3+3s^2+5s+15}$ **[7]**

(b) A linear system is represented by

$\dot{X}=\begin{bmatrix}-6 & 4\\ -2 & 0\end{bmatrix}X+\begin{bmatrix}1\\1\end{bmatrix}U;\qquad Y=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}X$

(i) Find the complete solution for Y(t) when U(t)=1(t), $X_1(0)=1, X_2(0)=0$

(ii) Draw a block diagram representing the system. **[5+3]**

12. Discuss the state controllability of the system

$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\end{bmatrix}=\begin{bmatrix}-3 & 1\\ -2 & 1.5\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix}+\begin{bmatrix}1\\4\end{bmatrix}u$

Prove the conditions used. **[3+4]**

13. If a continuous-time state equation is represented in discrete form as

X[(K+1)T]=G(T)X(KT) + H(T) U(KT)

Deduce the expressions for the matrices G(T) & H(T)

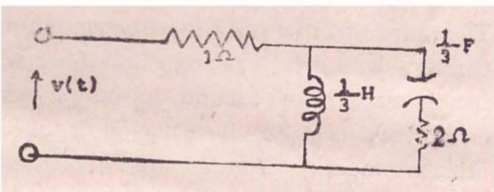
Discretise the continuous-time system described by

$\dot{X}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}X+\begin{bmatrix}0\\1\end{bmatrix}U;$

Assume the sampling period is 2 secs. **[5+4]**

14.(a) Choosing x_1=current through the inductor **[8]**

x_2=voltage across capacitor,determine the state equation for the system shown in fig below



(b) Explain controllability and observability. **[8]**

15. A linear system is represented by

$\dot{x}=\begin{bmatrix}-6 & 4\\ -2 & 0\end{bmatrix}x+\begin{bmatrix}1\\1\end{bmatrix}u$

$$Y=\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}x$$

(a)Find the complete solution for y(t) when

U(t)=1(t), x_1(0)=1 and x_2(0)=0

(b) Determine the transfer function

(c) Draw a block diagram representing the system **[9+4+3]**

16.(a) Derive an expression for the transfer function of a pump controlled hydraulic system. State the assumption made. **[8]**

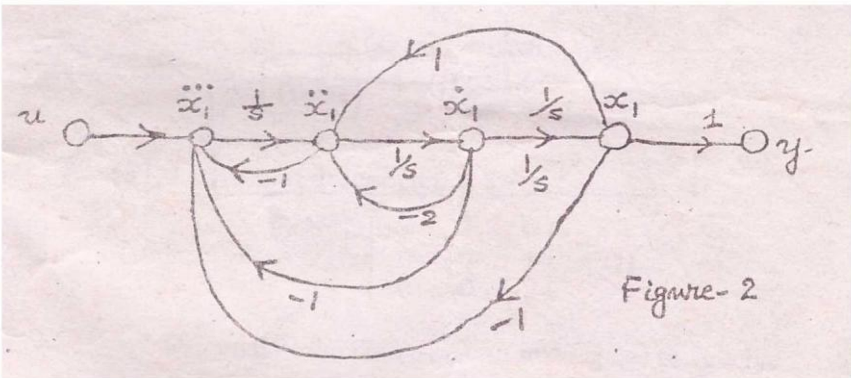
(b) Simulate a pneumatic PID controller and obtain its linearized transfer function. **[8]**

17. Describe the constructional features of a rate gyro, explain its principle of operation and obtain its transfer function. **[8]**

18. (a) Explain how poles of a closed loop control system can be placed at the desired points on the s plane. **[4]**

(b) Explain how diagonalisation of a system matrix a helps in the study of controllability of control systems. **[4]**

19. Construct the state space model of the system whose signal flow graph is shown in fig 2. **[7]**



20. (a) Define state of a system, state variables, state space and state vector. What are the advantages of state space analysis? **[5]**

(b) A two input two output linear dynamic system is governed by

$$\dot{X}(t)=\begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix}X(t)+\begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}R(t)$$

$$Y(t)=\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}X(t)$$

i)Find out the transfer function matrix. **[5]**

ii)Assuming $X(0) = 0$ find the output response Y(t) if **[5]**

$$R(t)=\begin{bmatrix}0\\e^{-3t}\end{bmatrix} \text{ for } t\geq 0$$

21.(a) A system is described by **[8]**

$$\dot{X}(t)=\begin{bmatrix}-4 & 1 & 0\\0 & -3 & 1\\0 & 0 & -2\end{bmatrix}X(t)$$

Diagonalise the above system making use of suitable transformation X=PZ

(b) Show how can you compute e^{At} using results of (a) **[7]**

22. Define controllability and observability and of control systems. **[4]**

23. A feed back system has a closed loop transfer function:

$$\frac{Y(s)}{R(s)} = \frac{10s + 40}{s^3 + s^2 + 3s}$$

Construct three different state models showing block diagram in each case. **[5×3]**

24. Explain the method of pole placement by state-feedback. Find the matrix $k=[k_1\ k_2]$ which is called the state feedback gain matrix for the closed loop poles to be located at $-1.8 \pm j2.4$ for the original system governed by the state equation:

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$$ **[6]**

25.(a) From a system represented by the state equation

$$\dot{x(t)} = A\, x(t)$$

The response of

$X(t)=\begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ when $x(0)=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

And $x(t)=\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$ when $x(o)=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Determine the system matrix A and the states transition matrix φ(t). **[12]**

(b) Prove non uniqueness of state space model. **[4]**

26.(a) Show the following system is always controllable

$$\dot{x(t)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

(b) Explain the design of state observer.

(c) Illustrate and explain pole placement by state feedback. **[4+4+4]**

MODULE-II

COMPENSATOR DESIGN

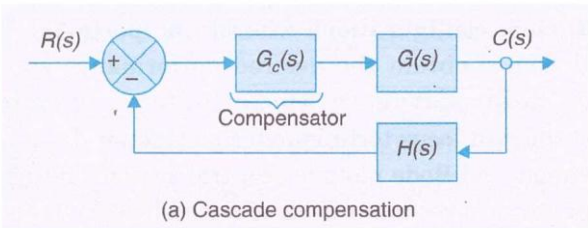
Every control system which has been designed for a specific application should meet certain performance specification. There are always some constraints which are imposed on the control system design in addition to the performance specification. The choice of a plant is not only dependent on the performance specification but also on the size , weight & cost. Although the designer of the control system is free to choose a new plant, it is generally not advised due to the cost & other constraints. Under this circumstances it is possible to introduce some kind of corrective sub-systems in order to force the chosen plant to meet the given specification. We refer to these sub-systems as **compensator** whose job is to compensate for the deficiency in the performance of the plant.

REALIZATION OF BASIC COMPENSATORS

Compensation can be accomplished in several ways.

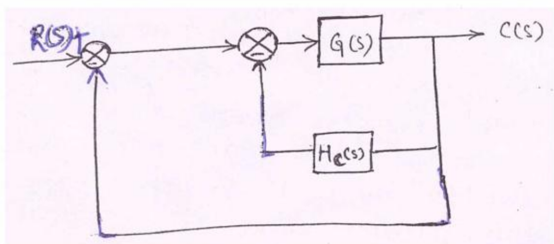
Series or Cascade compensation

Compensator can be inserted in the forward path as shown in fig below. The transfer function of compensator is denoted as $G_c(s)$, whereas that of the original process of the plant is denoted by G(s).

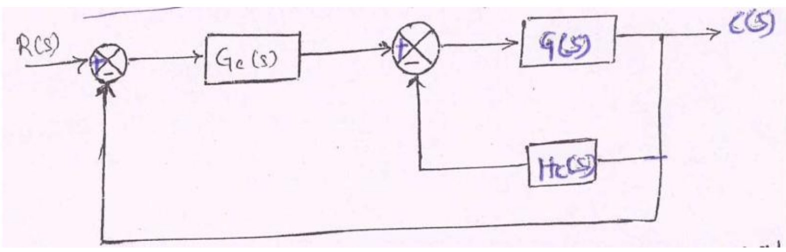


(a) Cascade compensation

feedback compensation



Combined Cascade & feedback compensation



Compensator can be electrical, mechanical, pneumatic or hydrolic type of device. Mostly electrical networks are used as compensator in most of the control system. The very simplest of these are Lead, lag & lead-lag networks.

Lead Compensator

Lead compensator are used to improve the transient response of a system.

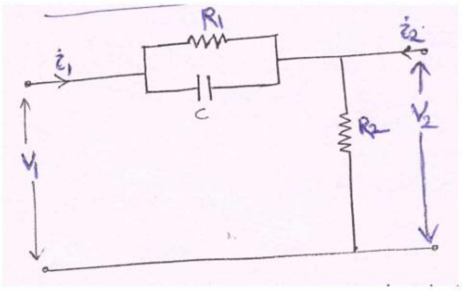


Fig: Electric Lead Network

Taking i_2=0 & applying Laplace Transform, we get

$$\frac{V_2(s)}{V_1(s)} = \frac{R_2(R_1Cs + 1)}{R_2 + R_2R_1Cs + R_1}$$

Let $\tau = R_1C$, $\alpha = \frac{R_2}{R_1+R_2} < 1$

$\frac{V_2(s)}{V_1(s)} = \frac{\alpha(\tau s+1)}{(1+\tau\alpha s)}$ Transfer function of Lead Compensator

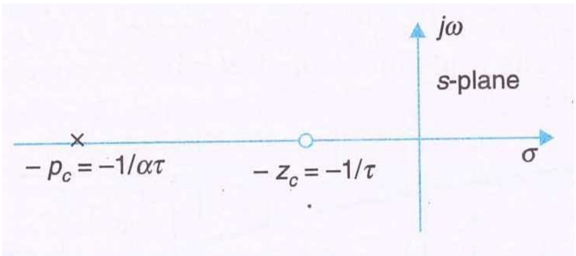


Fig: S-Plane representation of Lead Compensator

Bode plot for Lead Compensator

Maximum phase lead occurs at $w_m = \frac{1}{\tau\sqrt{\alpha}}$

Let ϕ_m = maximum phase lead

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

Magnitude at maximum phase lead $|G_c(jw)| = \frac{1}{\sqrt{\alpha}}$

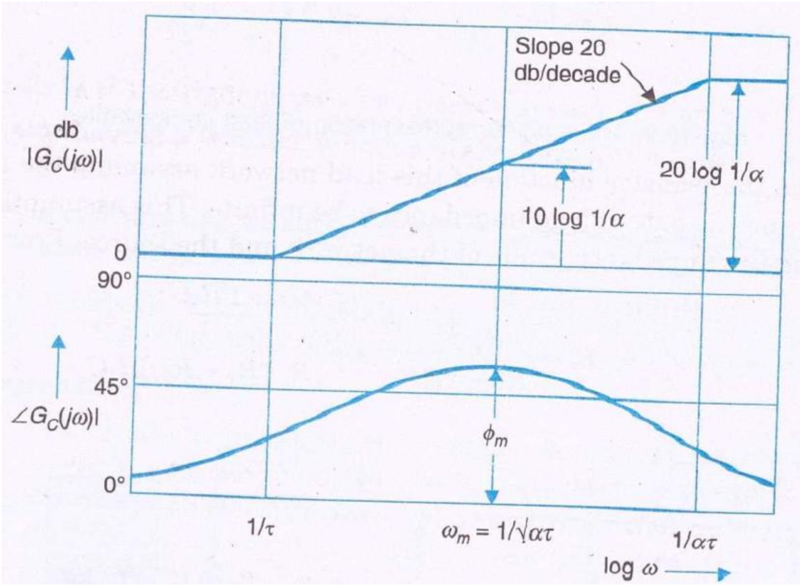


Fig: Bode plot of Phase Lead network with amplifier of gain $A = {}^{1}/_{\alpha}$

Lag Compensator

Lag compensator are used to improve the steady state response of a system.

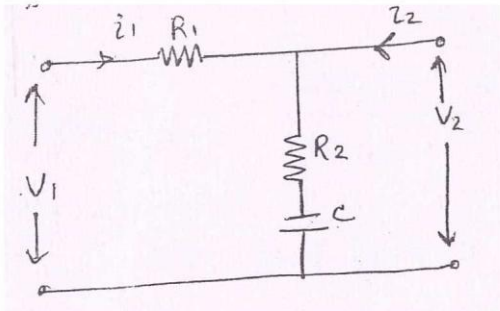


Fig: Electric Lag Network

Taking i_2=0 & applying Laplace Transform, we get

$$\frac{V_2(s)}{V_1(s)} = \frac{R_2Cs + 1}{(R_2 + R_1)Cs + 1}$$

Let $\tau = R_2C$, $\beta = \frac{R_1+R_2}{R_2} > 1$

$\frac{V_2(s)}{V_1(s)} = \frac{\tau s+1}{1+\tau\beta s}$ Transfer function of Lag Compensator

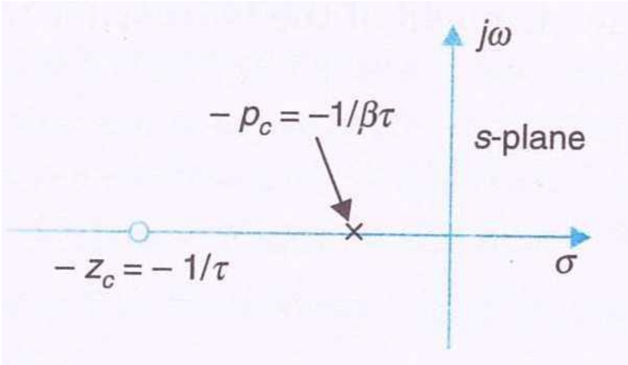


Fig: S-Plane representation of Lag Compensator

Bode plot for Lag Compensator

Maximum phase lag occurs at $w_m = \frac{1}{\tau\sqrt{\beta}}$

Let ϕ_m = maximum phase lag

$$\sin\phi_m = \frac{1-\beta}{1+\beta}$$

$$\beta = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

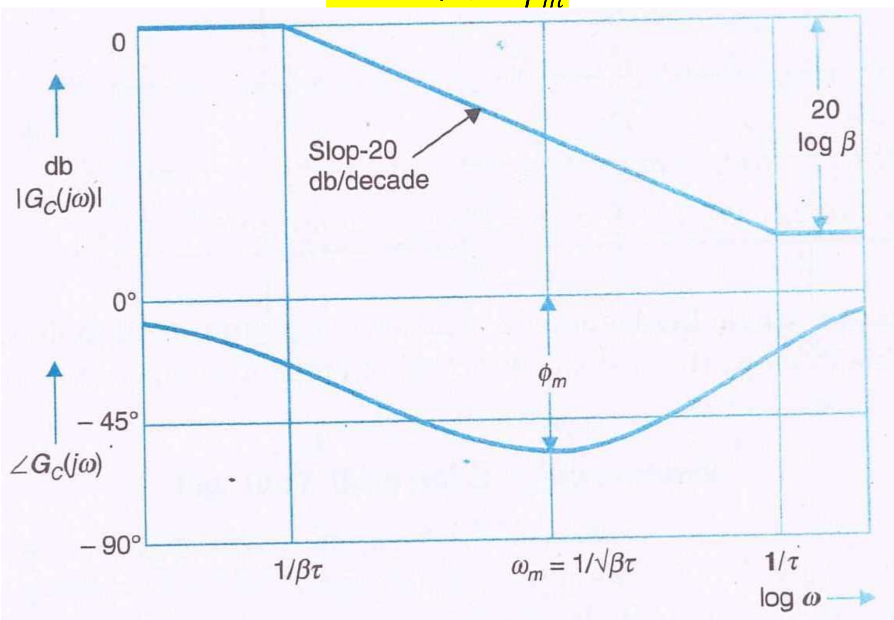


Fig: Bode plot of Phase Lag network

Cascade compensation in Time domain

Cascade compensation in time domain is conveniently carried out by the root locus technique. In this method of compensation, the original design specification on dynamic response are converted into ζ & w_n of a pair of desired complex conjugate closed loop pole based on the assumption the system would be dominated by these two complex pole therefore its dynamic behavior can be approximated by that of a second order system.

A compensator is now designed so that the least damped complex pole of the resulting transfer function correspond to the desired dominant pole & all other closed loop poles are located very close to the open loop zeros or relatively far away from the jw axis. This ensures that the poles other than the dominant poles make negligible contribution to the system dynamics.

Lead Compensation

- Consider a unity feedback system with a forward path unalterable Transfer function $G_f(s)$, then let the dynamic response specifications are translated into desired location S_d for the dominant complex closed loop poles.
- If the angle criteria as S_d is not meet i.e $\angle G_f(s) \neq \pm 180°$ the uncompensated Root Locus with variable open loop gain will not pass through the desired root location, indicating the need for the compensation.
- The lead compensator $G_c(s)$ has to be designed that the compensated root locus passes through S_d . In terms of angle criteria this requires that

$$\angle G_c(s_d)G_f(s_d) = \angle G_c(s_d) + \angle G_f(s_d) \pm 180°$$

$$\angle G_c(s_d) = \phi = \pm 180° - \angle G_f(s_d)$$

- Thus for the root locus for the compensated system to pass through the desired root location the lead compensator pole-zero pair must contribute an angle ϕ .
- For a given angle ϕ required for lead compensation there is no unique location for pole-zero pair. The best compensator pole-zero location is the one which gives the largest value of .

  Where $\alpha = \frac{Z_c}{P_c}$

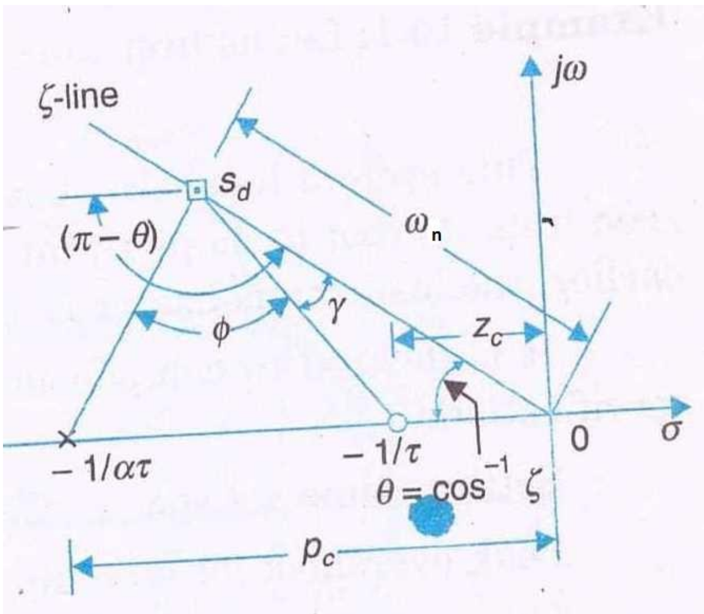


Fig: Angle contribution of Lead compensator

- The compensator zero is located by drawing a line from S_d making an angle γ with ζ line.
- The compensator pole then located by drawing a further requisite angle ϕ to be contribute at S_d by the pole zero pair. From the geometry of the figure

$$\frac{z_c}{\sin\gamma} = \frac{w_n}{\sin(\pi-\theta-\gamma)}$$

$$\Rightarrow z_c = \frac{w_n \sin\gamma}{\sin(\pi-\theta-\gamma)}$$

Assuming big triangle

$$\frac{p_c}{\sin(\phi+\gamma)} = \frac{w_n}{\sin(\pi-\theta-\gamma-\phi)}$$

$$\Rightarrow p_c = \frac{w_n \sin(\phi+\gamma)}{\sin(\pi-\theta-\gamma-\phi)}$$

$$\alpha = \frac{Z_c}{P_c} == \frac{\sin(\pi-\theta-\gamma-\phi)\sin\gamma}{\sin(\pi-\theta-\gamma)\sin(\phi+\gamma)}$$

To find α_{max} , $\frac{d\alpha}{d\gamma} = 0$

$$\Rightarrow \gamma = \frac{1}{2}(\pi-\theta-\phi)$$

Though the above method of locating the lead compensator pole-zero yields the largest value of α, it does not guarantee the dominance of the desired closed loop poles in the compensated root-locus. The dominance condition must be checked before completing the design. With compensator pole-zero so located the system gain at S_d is computed to determine the error constant. If the value of the error constant so obtained is unsatisfactory the above procedure is repeated after readjusting the compensator pole-zero location while keeping the angle contribution fixed as ϕ.

Lag Compensation

Consider a unit feedback system with forward path transfer function

$$G_f(s) = \frac{k\prod_{i=1}^{m}(s+z_i)}{s^r\prod_{j=r+1}^{n}(s+p_j)}$$

At certain value of K, this system has satisfactory transient response i.e its root locus plot passes through(closed to) the desired closed loop poles location S_d .

It is required to improve the system error constant to a specified value K_e^c without damaging its transient response. This requires that after compensation the root locus should continue to pass through S_d while the error constant at S_d is raised to K_e^c. To accomplish this consider adding a lag compensator pole-zero pair with zero the left of the pole. If this pole-zero pair is located closed to each other it will contribute a negligible angle at S_d such that S_d continues to lie on the root locus of the compensated system.

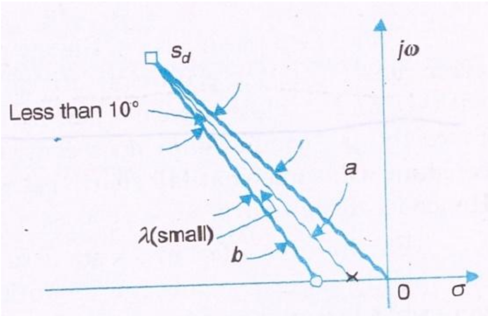


Fig:Locating the Lag Compensator Pole-zero

From the above fig. that apart from being close to each other the pole-zero pair close to origin, the reason which will become obvious from discussion below.
The gain of the uncompensated system at S_d is given by

$$K^{uc}(s_d) = \frac{|s_d|^r \prod_{j=r+1}^{n}(s_d + p_j)}{\prod_{i=1}^{m}(s_d + z_i)}$$

For compensated system the system gain at S_d is given by

$$K^{c}(s_d) = \frac{|s_d|^r \prod_{j=r+1}^{n}(s_d + p_j)}{\prod_{i=1}^{m}(s_d + z_i)} \quad \frac{a}{b}$$

Since the pole & zero are located close to each other they are nearly equidistance from S_d
i.e $a \approx b$
i.e $K^{c}(s_d) \cong K^{uc}(s_d)$

The error constant for the compensated system is given by

$$K_e^c = K^c(s_d) \; \frac{\prod_{i=1}^{m} z_i}{\prod_{j=r+1}^{n} p_j} \; \frac{z_c}{p_c}$$

$$K_e^c \cong K^{uc}(s_d) \; \frac{\prod_{i=1}^{m} z_i}{\prod_{j=r+1}^{n} p_j} \; \frac{z_c}{p_c}$$

$$K_e^c = K_e^{uc} \; \frac{z_c}{p_c}$$

K_e^{uc} → is error constant at S_d for uncompensated system.
K_e^c → is error constant at S_d for compensated system.

$$\beta = \frac{z_c}{p_c} = \frac{K_e^c}{K_e^{uc}} \dots\dots\dots\dots\dots..(1)$$

The β parameter of lag compensator is nearly equal to the ratio of specified error constant to the error constant of the uncompensated system.
Any value of $\beta = \frac{z_c}{p_c} > 1$ with $-p_c$ & $-z_c$ close to each other can be realized by keeping the pole-zero pair close to origin.

Since the Lag compensator does contribute a small negative angle λ at S_d , the actual error constant will some what fall short of the specified value if β obtained from equation(1) is used. Hence for design purpose we choose β somewhat larger than that the given by this equation(1).
For the effect of the small lag angle λ is to give the closed loop pole S_d with specified ζ but slightly lower w_n . This can be anticipated & counteracted by taking the w_n of S_d to be somewhat larger than the specified value.

Cascade compensation in Frequency domain

Lead Compensation

Procedure of Lead Compensation
Step1: Determine the value of loop gain K to satisfy the specified error constant. Usually the error constant (K_p,K_v,K_a) & Phase margin are the specification given.
Step2: For this value of K draw the bode plot & determine the phase margin ϕ for the system.
Step3: If ϕ_s =specified phase margin &
ϕ = phase margin of uncompensated system (found out from the bode plot drawn)
ϵ =margin of safety (since crossover frequency may increase due to compensation)

- ϵ is the unknown reduction in phase angle $\angle G_f(s)$ on account of the increase in cross-over frequency. A guess is made on the value of ϵ depending on the slope in this region of the dB-log w plot of the uncompensated system.
- For a slope of -40dB/decade $\epsilon = 5° - 10°$ is a good guess. The guess value may have to be as high as $15°$ *to* $20°$ for a slope of -60dB/decade.
- Phase lead required $\phi_l = \phi_s - \phi + \epsilon$

Step4: Let $\phi_l = \phi_m$
Determine

$$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m}$$

If $\phi_m > 60°$, two identical networks each contributing a maximum lead of $\phi_l/2$ are used.

Step5: Find the frequency ω_m at which the uncompensated system will have a gain equals to $-10\log\frac{1}{\alpha}$ from the bode plot drawn.

Take $\omega_{c2} = \omega_m$ = cross-over frequency of compensated system.

Step6: Corner frequency of the network are calculated as

$$\omega_1 = \frac{1}{\tau} = \omega_m\sqrt{\alpha} \quad ,\omega_2 = \frac{1}{\tau\alpha} = \frac{\omega_m}{\sqrt{\alpha}}$$

Transfer function for compensated system in Lead network $G_c(s) = \frac{s+\frac{1}{\tau}}{s+\frac{1}{\tau\alpha}}$

Step7: Draw the magnitude & Phase plot for the compensated system & check the resulting phase margin. If the phase margin is still low raise the value of ϵ & repeat the procedure.

Lag Compensation

Procedure of Lead Compensation

Step1: Determine the value of loop gain K to satisfy the specified error constant.

Step2: For this value of K draw the bode plot & determine the phase margin ϕ for the system.

Step3: If ϕ_s =specified phase margin &

ϕ = phase margin of uncompensated system (found out from the bode plot drawn)

ϵ =margin of safety ($5° - 10°$)

- For a suitable ϵ find $\phi_2 = \phi_s + \epsilon$,where ϕ_2 is measured above $-180°$ line.

Step4: Find the frequency ω_{c2} where the uncompensated system makes a phase margin contribution of ϕ_2.

Step5: Measure the gain of uncompensated system at ω_{c2}. Find β from the equation

$$gain\ at\ \omega_{c2} = 20\log\beta$$

Step6: Choose the upper corner frequency $\omega_2 = \frac{1}{\tau}$ of the network one octave to one decade below $\omega_{c2}(i.e\ between\ \frac{\omega_{c2}}{2}\ to\ \frac{\omega_{c2}}{10})$

Step7: Thus β & τ are determined which can be used to find the transfer function of Lag compensator.

$$G_c(s) = \frac{1}{\beta}\left[\frac{s+\frac{1}{\tau}}{s+\frac{1}{\tau\alpha}}\right]$$

Compensated Transfer function $G(s) = G_f\ G_c$

Draw the bode plot of the compensated system & check if the given specification are met.

MATLab Code

Plotting rootlocus with MATLAB(**rlocus**)

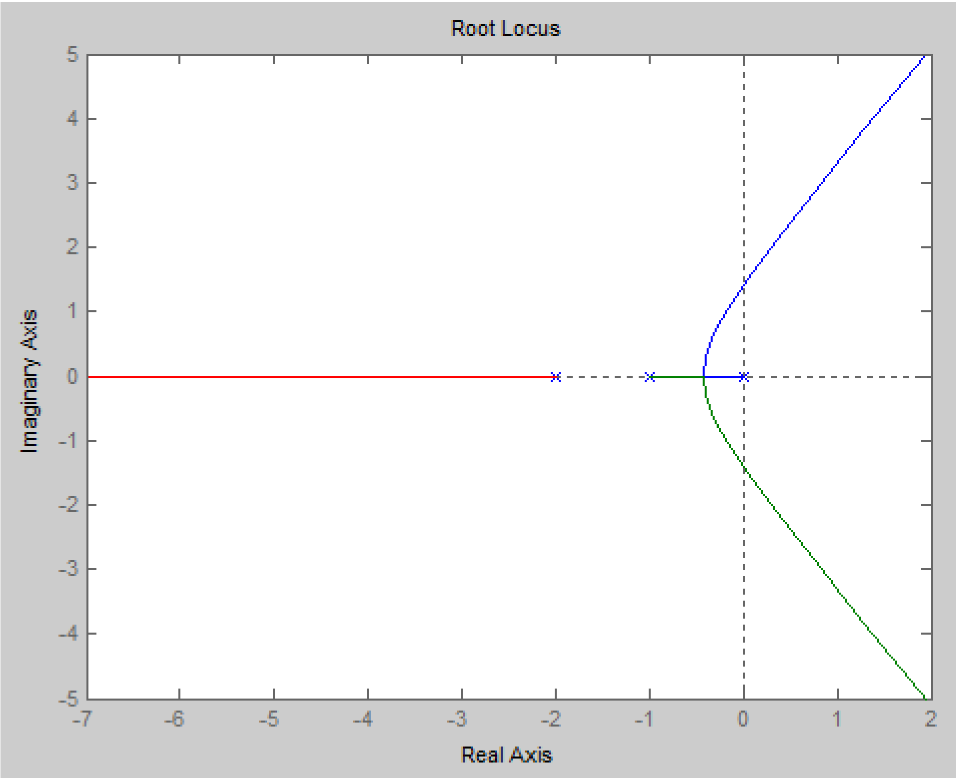
Consider a unity-feedback control system with the following feedforward transfer function:

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Using MATLAB, plot the rootlocus.

$$G(s) = \frac{K}{s(s+1)(s+2)} = \frac{K}{s^3 + 3s^2 + 2s}$$

```
num=[1];
den=[1 3 2 0];
h = tf(num,den);
rlocus(h)
```



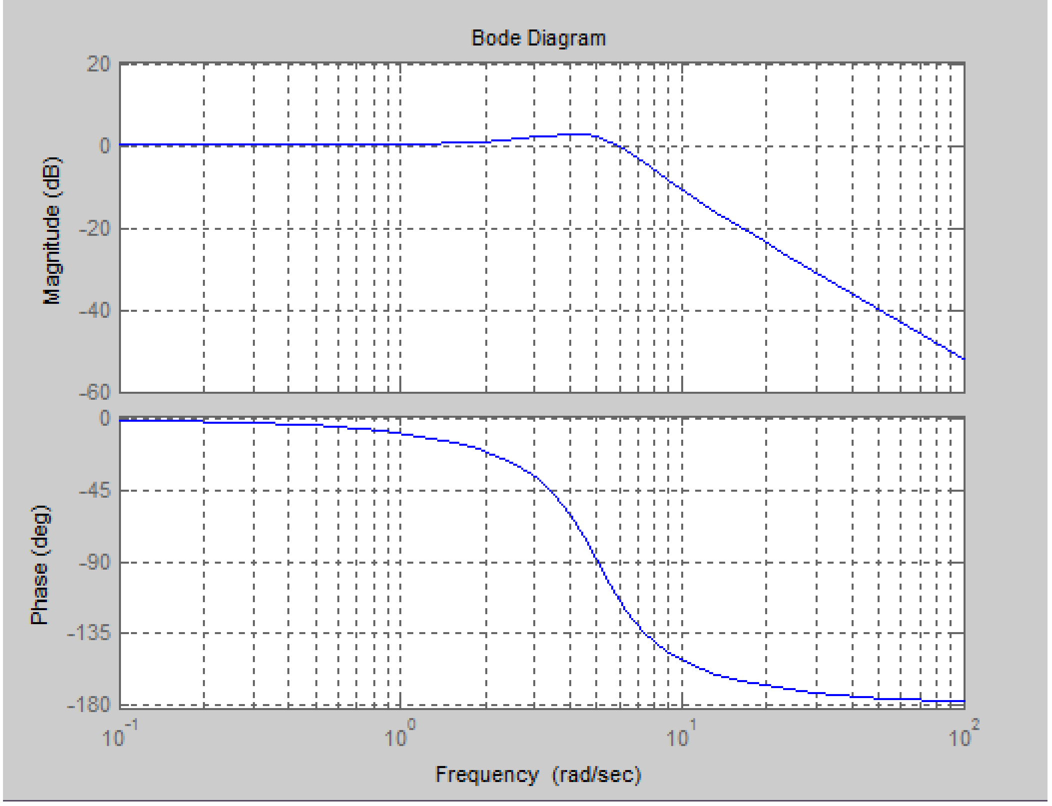
Plotting Bode Diagram with MATLAB(**bode**)

Consider the following transfer function

$$G(s) = \frac{25}{s^2 + 4s + 25}$$

Plot the Bode diagram for this transfer function

```
num=[25];
den=[1 4 25];
bode(num,den)
grid on
```



Digital Control Systems

Sampling operation in sampled data and digital control system is used to model either the sample and hold operation or the fact that the signal is digitally coded. If the sampler is used to represent S/H (Sample and Hold) and A/D (Analog to Digital) operations, it may involve delays, finite sampling duration and quantization errors. On the other hand if the sampler is used to represent digitally coded data the model will be much simpler. Following are two popular sampling operations:

1. Single rate or periodic sampling
2. Multi-rate sampling

We would limit our discussions to periodic sampling only.

1.1 Finite pluse width sampler

In general, a sampler is the one which converts a continuous time signal into a pulse modulated or discrete signal. The most common type of modulation in the sampling and hold operation is the pulse amplitude modulation.

The symbolic representation, block diagram and operation of a sampler are shown in Figure 1. The pulse duration is *p* second and sampling period is *T* second. Uniform rate sampler is a linear device which satisfies the principle of superposition. As in Figure 1, *p*(t) is a unit pulse train with period *T*.

$$p(t) = \sum_{k=-\infty}^{\infty} [u_s(t-kT) - u_s(t-kT-p)]$$

where $u_s(t)$ represents unit step function. Assume that leading edge of the pulse at $t = 0$ coincides with $t = 0$. Thus $f_p^*(t)$ can be written as

$$f_p^*(t) = f(t) \sum_{k=-\infty}^{\infty} [u_s(t-kT) - u_s(t-kT-p)]$$

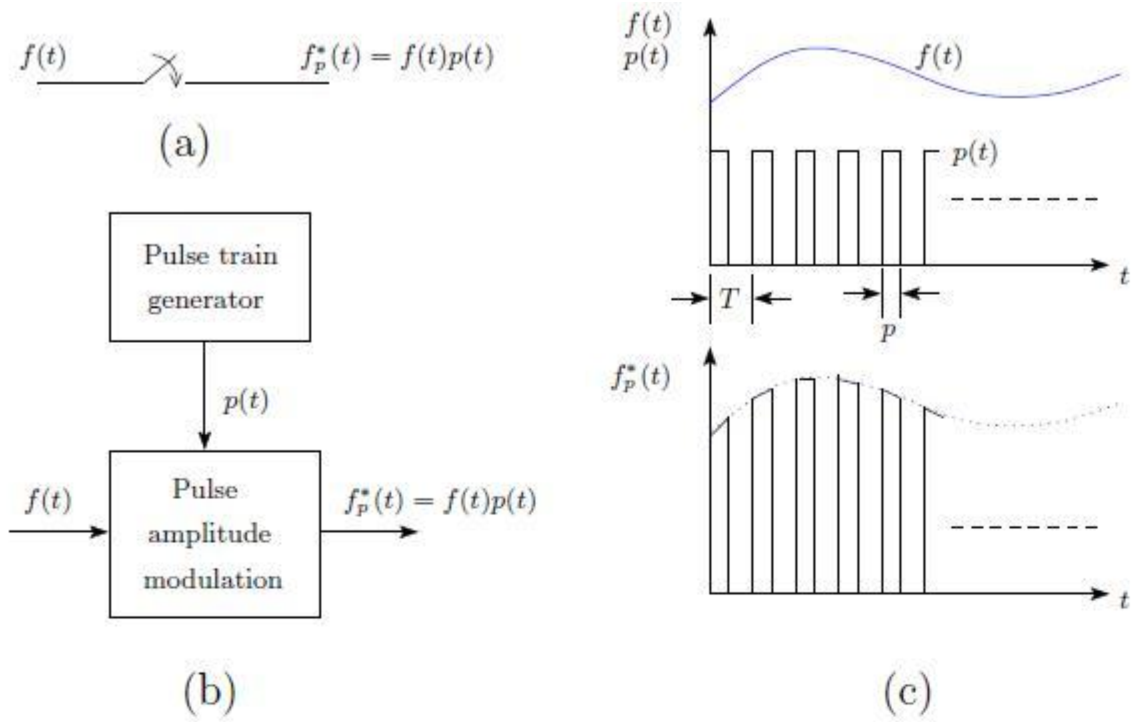


Figure : Finite pulse with sampler : (a) Symbolic representation (b) Block diagram (c) Operation

According to Shannon's sampling theorem, "if a signal contains no frequency higher than w_c rad/sec, it is completely characterized by the values of the signal measured at instants of time separated by $T = \pi/w_c$ sec."

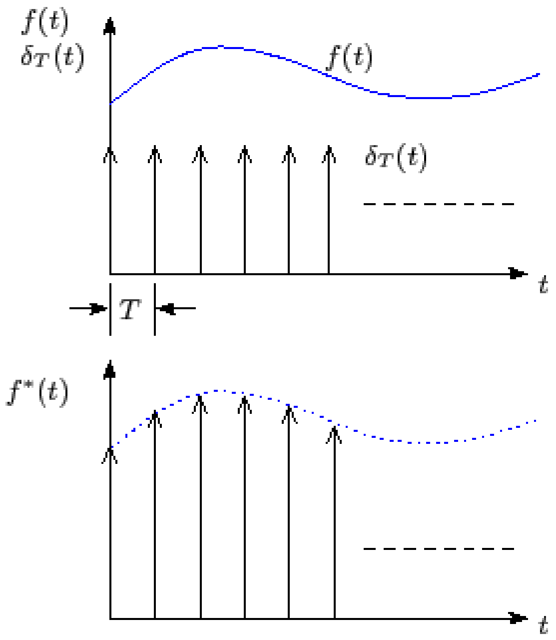
Sampling frequency rate should be greater than the **Nyquist rate** which is twice the highest frequency component of the original signal to avoid aliasing.

If the sampling rate is less than twice the input frequency, the output frequency will be different from the input which is known as **aliasing**. The output frequency in that case is called **alias frequency**and the period is referred to as **alias period**.

The overlapping of the high frequency components with the fundamental component in the frequency spectrum is sometimes referred to as **folding** and the frequency $w_s/2$ is often known as **folding frequency**. The frequency w_c is called **Nyquist frequency**.

A low sampling rate normally has an adverse effect on the closed loop stability. Thus, often we might have to select a sampling rate much higher than the theoretical minimum.

Ideal Sampler : In case of an ideal sampler, the carrier signal is replaced by a train of unit impulses as shown in Figure 2. The sampling duration p approaches 0, i.e., its operation is instantaneous.



the output of an ideal sampler can be expressed as

$$f^*(t) = \sum_{k=0}^{\infty} f(kT)\delta(t-kT)$$

$$\Rightarrow F^*(s) = \sum_{k=0}^{\infty} f(kT)e^{-kTs}$$

One should remember that practically the output of a sampler is always followed by a hold device which is the reason behind the name sample and hold device. Now, the output of a hold device will be the same regardless the nature of the sampler and the attenuation factor *p* can be dropped in that case. Thus the sampling process can be be always approximated by an ideal sampler or impulse modulator.

Z- Transform

Let the output of an ideal sampler be denoted by $f^*(t)$

$$L[f^*(t)] = f^*(s) = \sum_{K=0}^{\infty} f(KT)\, e^{-KTs}$$

If we substitute $Z = e^{Ts}$,then we get $F(z)$, is the Z-transform of $f(t)$ at the sampling instants k

$$F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k}$$

Z - Transforms of some elementary functions

Unit step function is defined as:

$$u_s(t) = 1, \quad \text{for} \quad t \geq 0$$
$$= 0, \quad \text{for} \quad t < 0$$

Assuming that the function is continuous from right

$$X(z) = \sum_{k=0}^{\infty} u_s(kT) z^{-k}$$
$$= \sum_{k=0}^{\infty} z^{-k}$$
$$= 1 + z^{-1} + z^{-2} + z^{-3} + \ldots\ldots\ldots$$
$$= \frac{1}{1 - z^{-1}}$$
$$= \frac{z}{z - 1}$$

The above series converges if $|Z| > 1$

Unit ramp function is defined as:

$$u_r(t) = t, \quad \text{for} \quad t \geq 0$$
$$= 0, \quad \text{for} \quad t < 0$$

The Z-transform is:

$$U_r(z) = \frac{Tz}{(z-1)^2}$$
$$= T\frac{z^{-1}}{(1 - z^{-1})^2}$$

The above series converges if $|Z| > 1$

For a **polynomial function $x(k) = a^k$**

The Z-transform is:

$$X(z) = \frac{1}{1 - a.z^{-1}}$$
$$= \frac{z}{z - a}$$

With ROC: $|Z| > a$

Exponential function is defined as:

$$x(t) = e^{-at}, \quad \text{for} \quad t \geq 0$$
$$= 0, \quad \text{for} \quad t < 0$$

We have $x(kT) = e^{-akT}$ for $k = 0, 1, 2 \cdots$. Thus,

$$X(z) = \frac{1}{1 - e^{-aT} z^{-1}}$$
$$= \frac{z}{z - e^{-aT}}$$

Properties of Z-transform

Multiplication by a constant: $Z[ax(t)] = aX(z)$, where $X(z) = Z[x(t)]$.

Linearity: If $x(k) = \alpha f(k) \pm \beta g(k)$, then $X(z) = \alpha F(z) \pm \beta G(z)$.

Multiplication by a^k: $Z[a^k x(k)] = X(a^{-1}z)$

Realshifting: $Z[x(t - nT)] = z^{-n}X(z)$ and

$$z[x(t + nT)] = z^n \left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k} \right]$$

Complex shifting: $Z[e^{\pm at}x(t)] = X(ze^{\mp aT})$

Initial value theorem:

$$x(0) = \lim_{z \to \infty} X(z)$$

Final value theorem:

$$\lim_{k \to \infty} x(k) = \lim_{z \to 1} [(1 - z^{-1})X(z)]$$

Inverse Z-transforms

f(t) is the continuous time function whose Z-transform is *F*(z). Then the inverse transform is not necessarily equal to *f*(t), rather it is equal to *f*(*k*T) which is equal to *f*(t) only at the sampling instants. Once *f*(t) is sampled by an the ideal sampler, the information between the sampling instants is totally lost and we cannot recover actual *f*(t) from *F*(z).

$$\Rightarrow f(kT) = Z^{-1}[F(z)]$$

The transform can be obtained by using

1. Partial fraction expansion
2. Power series
3. Inverse formula.

The Inverse Z-transform formula is given as:

$$f(kT) = \frac{1}{2\pi j} \oint_{\Gamma} F(z) z^{k-1} dz$$

MATLab Code to Obtain the inverse Z transform (filter)

Example

Obtain the inverse z transform of $X(z) = \frac{z(z+2)}{(z-1)^2}$

X(z) can be written as

$$X(z) = \frac{z^2 + 2z}{z^2 - 2z + 1}$$

```
num=[1 2 0];
den=[1 -2 1];
u=[1 zeros(1,30)];%If the values of x(k) for k=0,1,2,....,30 are desired
filter(num,den,u)
```

```
ans =

  Columns 1 through 15

    1    4    7   10   13   16   19   22   25   28   31   34   37   40   43

  Columns 16 through 30

   46   49   52   55   58   61   64   67   70   73   76   79   82   85   88

  Column 31

   91
```

Application of Z-transform in solving Difference Equation

One of the most important applications of Z-transform is in the solution of linear difference equations. Let us consider that a discrete time system is described by the following difference equation.

$$y(k+2) + 0.5y(k+1) + 0.06y(k) = -(0.5)^{k+1}$$

The initial conditions are $y(0) = 0, y(1) = 0$.
We have to find the solution $y(k)$ for $k > 0$.
Taking z-transform on both sides of the above equation:

$$\begin{aligned} z^2Y(z) + 0.5zY(z) + 0.06Y(z) &= -0.5\frac{z}{z-0.5} \\ \text{or,}\ \ Y(z) &= -\frac{0.5z}{(z-0.5)(z^2+0.5z+0.06)} \\ &= -\frac{0.5z}{(z-0.5)(z+0.2)(z+0.3)} \end{aligned}$$

Using partial fraction expansion:

$$Y(z) = -\frac{0.8937z}{z-0.5} + \frac{7.143z}{z+0.2} - \frac{6.25z}{z+0.3}$$

Taking Inverse Laplace: $y(k) = -0.893(0.5)^k + 7.143(-0.2)^k - 6.25(-0.3)^k$

To emphasize the fact that $y(k) = 0$ for $k < 0$, it is a common practice to write the solution as:

$$y(k) = -0.893(0.5)^k u_s(k) + 7.143(-0.2)^k u_s(k) - 6.25(-0.3)^k u_s(k)$$

where $u_s(k)$ is the unit step sequence.

<u>Relationship between s-plane and z-plane</u>

In the analysis and design of continuous time control systems, the pole-zero configuration of the transfer function in s-plane is often referred. We know that:

. Left half of s-plane $\Rightarrow$ Stable region.

. Right half of s-plane $\Rightarrow$ Unstable region.

For relative stability again the left half is divided into regions where the closed loop transfer function poles should preferably be located.

Similarly the poles and zeros of a transfer function in z-domain govern the performance characteristics of a digital system.

One of the properties of $F^*(s)$ is that it has an infinite number of poles, located periodically with intervals of $\pm m w_s$ with $m = 0, 1, 2, \ldots\ldots$, in the s-plane where w_s is the sampling frequency in rad/sec.

If the primary strip is considered, the path, as shown in Figure below, will be mapped into a unit circle in the z-plane, centered at the origin.

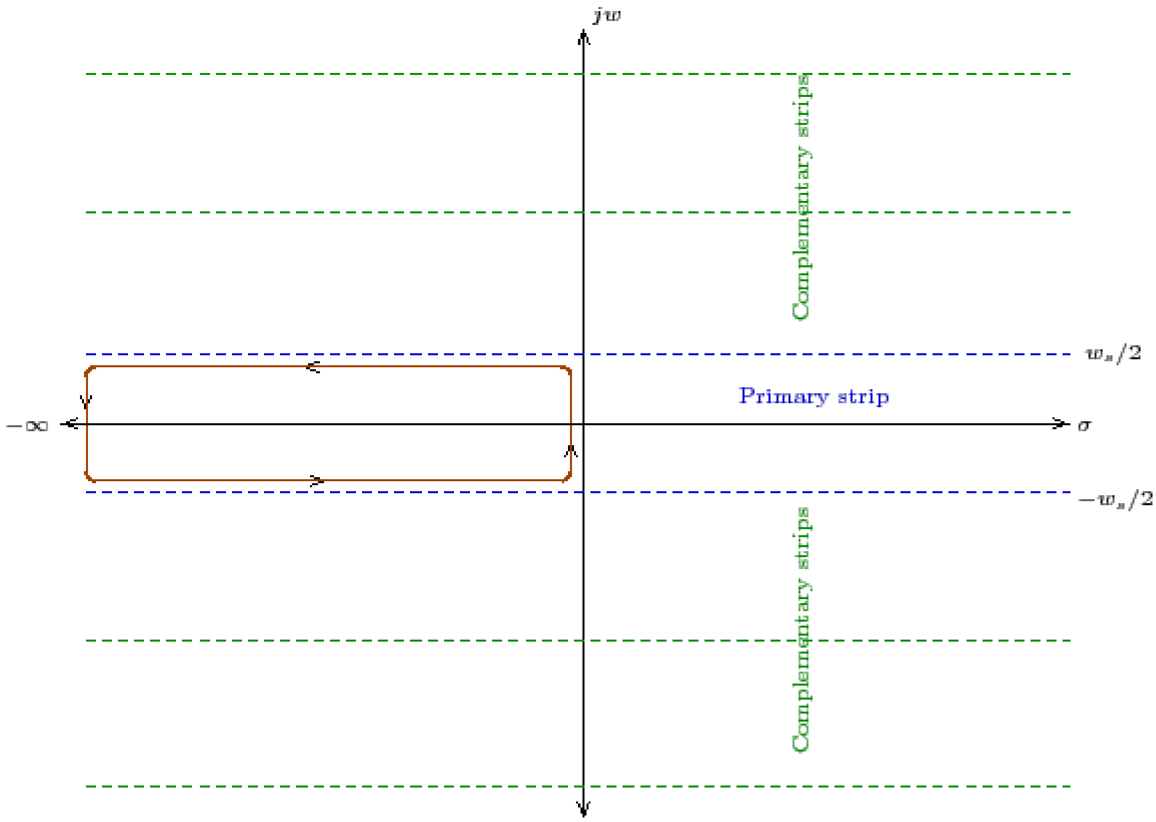


Figure : Primary and complementary strips in s-plane

The mapping is shown in Figure below.

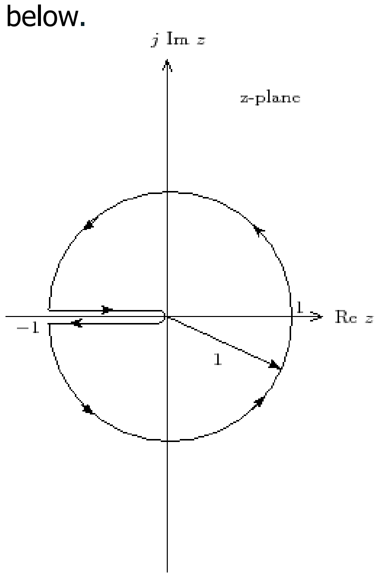


Figure : Mapping of primary strip in z-plane

Since

$$e^{(s+jmw_s)T} = e^{Ts}e^{j2\pi m}$$

$$= e^{Ts}$$

$$= z$$

where m is an integer, all the complementary strips will also map into the unit circle.

Mapping guidelines

1. All the points in the left half s-plane correspond to points inside the unit circle in z-plane.
2. All the points in the right half of the s-plane correspond to points outside the unit circle.
3. Points on the *jw* axis in the s-plane correspond to points on the unit circle $|Z| = 1$ in the z-plane.

$$s = jw$$

$$z = e^{Ts}$$

$$= e^{jwT} \Rightarrow \quad \text{magnitude} = 1$$

Pluse Transfer Function

Pulse transfer function relates Z-transform of the output at the sampling instants to the Z-transform of the sampled input.

When the same system is subject to a sampled data or digital signal ***r*(t)***, the corresponding block diagram is given in Figure 1 .

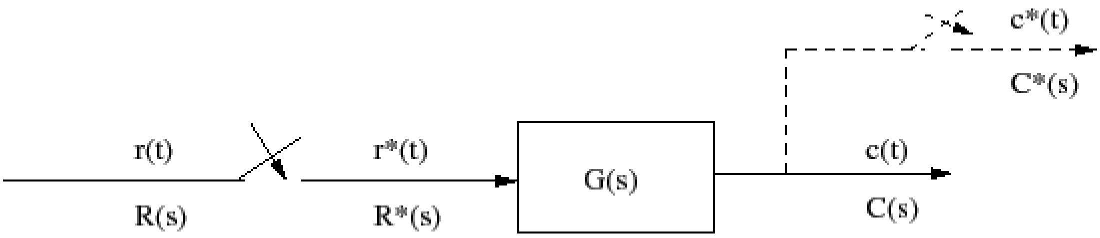


Figure 1: Block diagram of a system subject to a sampled input

The output of the system is ***C(s)*** = ***G(s)R*(s)***. The transfer function of the above system is difficult to manipulates because it contains a mixture of analog and digital components. Thus, for ease of manipulation, it is desirable to express the system characteristics by a transfer function that relates ***r*(t)*** to ***c*(t)***, a fictitious sampler output, as shown in Figure 1.

One can then write:

$$C^*(s) = \sum_{K=0}^{\infty} c(KT)\, e^{-KTs}$$

Since ***c(kT)*** is periodic,

$$C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s + jnw_s) \qquad with\ C(0) = 0$$

The detailed derivation of the above expression is omitted. Similarly,

$$R^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} R(s + jnw_s)$$

Again,

$$C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s + jnw_s)$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} R^*(s + jnw_s)G(s + jnw_s)$$

Since ***R*(s)*** is periodic ***R*(s + jnw$_s$)*** = ***R*(s)***. Thus

$$C^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} R^*(s)G(s + jnw_s)$$

$$= R^*(s)\frac{1}{T}\sum_{n=-\infty}^{\infty} G(s + jnw_s)$$

If we define

$G^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s + jnw_s)$, then $C^*(s) = R^*(s)G^*(s)$.

$$G^*(s) = \frac{C^*(s)}{R^*(s)}$$

is known as **pulse transfer function**. Sometimes it is also referred to as the **starred transfer function**.

If we now substitute **$z = e^{Ts}$** in the previous expression, we will directly get the **z-transfer functionG(z)** as

$$G(z) = \frac{C(z)}{R(z)}$$

Pulse transfer of discrete data systems with cascaded elements

1. Cascaded elements are separated by a sampler

The block diagram is shown in Figure below.

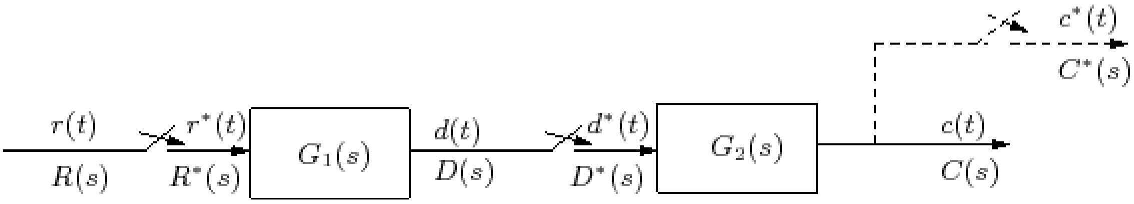


Figure: Discrete data system with cascaded elements, separated by a sampler

The input-output relations of the two systems G_1 and G_2 are described by

$$D(z) = G_1(z)R(z)$$

and

$$C(z) = G_2(z)D(z)$$

Thus the input-output relation of the overall system is

$$C(z) = G_1(z)G_2(z)R(z)$$

We can therefore conclude that the z-transfer function of two linear system separated by a sampler are the products of the individual z-transfer functions.

2. Cascaded elements are not separated by a sampler

The block diagram is shown in Figure below

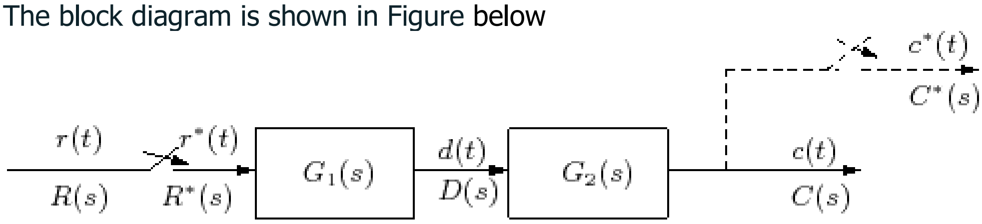


Figure : Discrete data system with cascaded elements, not separated by a sampler

The continuous output C(s) can be written as

$$C(s) = G_1(s)G_2(s)R^*(s)$$

The output of the fictitious sampler is

$$C(z) = Z\left[G_1(s)G_2(s)\right]R(z)$$

z-transform of the product $G_1(s)G_2(s)$s is denoted as

$$Z\left[G_1(s)G_2(s)\right] = G_1G_2(z) = G_2G_1(z)$$

Note:

$$G_1G_2(z) \neq G_1(z)G_2(z)$$

The overall output is thus,

$$C(z) = G_1G_2(z)R(z)$$

Pluse Transfer Function of Closed Loop Systems

A simple single loop system with a sampler in the forward path is shown in Figure below.

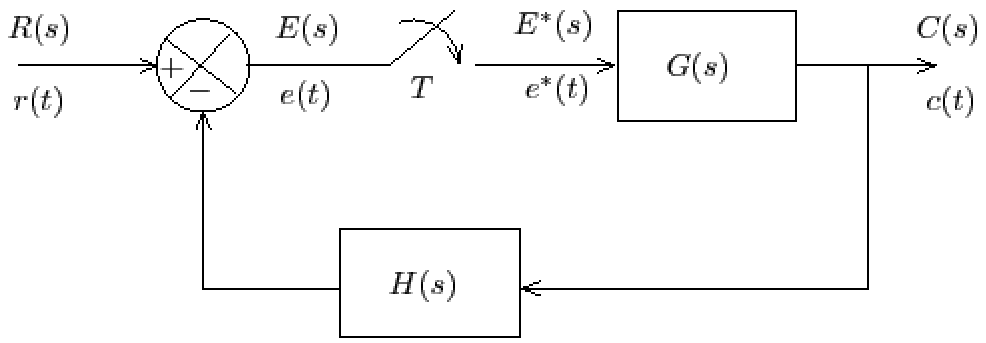


Figure : Block diagram of a closed loop system with a sampler in the forward path

The objective is to establish the input-output relationship. For the above system, the output of the sampler is regarded as an input to the system. The input to the sampler is regarded as another output. Thus the input-output relations can be formulated as

$$E(s) = R(s) - G(s)H(s)E^*(s)$$

$$C(s) = G(s)E^*(s)$$

Taking pulse transform on both sides of E(s)

$$E^*(s) = R^*(s) - GH^*(s)E^*(s) \qquad (3)$$

where

$$GH^*(s) = [G(s)H(s)]^*$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s + jnw_s)H(s + jnw_s)$$

We can write from equation (3),

$$E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$$

$$\Rightarrow C(s) = G(s)E^*(s)$$

$$= \frac{G(s)R^*(s)}{1 + GH^*(s)}$$

Taking pulse transformation on both sides of C(s)

$$C^*(s) = [G(s)E^*(s)]^*$$

$$= G^*(s)E^*(s)$$

$$= \frac{G^*(s)R^*(s)}{1 + GH^*(s)}$$

$$\therefore \frac{C^*(s)}{R^*(s)} = \frac{G^*(s)}{1 + GH^*(s)}$$

$$\Rightarrow \frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

where $GH(z) = Z[G(s)H(s)]$.

Now, if we place the sampler in the feedback path, the block diagram will look like the Figure 2.

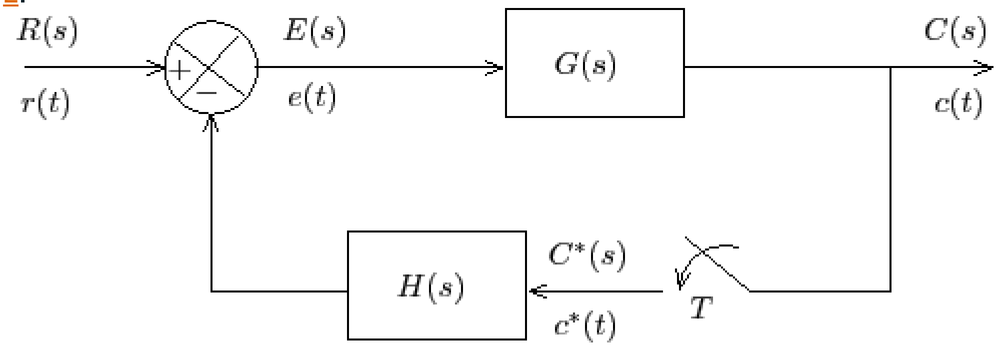


Figure 2: Block diagram of a closed loop system with a sampler in the feedback path

The corresponding input output relations can be written as:

$$E(s) = R(s) - H(s)C^*(s) \qquad \ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

$$C(s) = G(s)E(s) = G(s)R(s) - G(s)H(s)C^*(s) \qquad \ldots\ldots\ldots\ldots\ldots(5)$$

Taking pulse transformation of equations (4) and (5)

$$E^*(s) = R^*(s) - H^*(s)C^*(s)$$

$$C^*(s) = GR^*(s) - GH^*(s)C^*(s)$$

where, $$GR^*(s) = [G(s)R(s)]^*$$

$$GH^*(s) = [G(s)H(s)]^*$$

can be written as

$$C^*(s) = \frac{GR^*(s)}{1 + GH^*(s)}$$

$$\Rightarrow C(z) = \frac{GR(z)}{1 + GH(z)}$$

We can no longer define the input output transfer function of this system by either $\frac{C^*(s)}{R^*(s)}$ or $\frac{C(z)}{R(z)}$.

Since the input r(t) is not sampled, the sampled signal $r^*(t)$ does not exist.

The continuous-data output C(s) can be expressed in terms of input as.

$$C(s) = G(s)R(s) - \frac{G(s)H(s)}{1 + GH^*(s)} GR^*(s)$$

Stability Analysis of closed loop system in z-plane

Similar to continuous time systems, the stability of the following closed loop system

$$\frac{C(z)}{R(Z)} = \frac{G(z)}{1 + GH(z)}$$

can also be determined from the location of closed loop poles in z-plane which are the roots of the characteristic equation

$$1 + GH(z) = 0$$

1.For the system to be stable, the closed loop poles or the roots of the characteristic equation must lie within the unit circle in z-plane. Otherwise the system would be unstable.

2. If a simple pole lies at $|z| = 1$, the system becomes marginally stable. Similarly if a pair of complex conjugate poles lie on the $|z| = 1$ circle, the system is marginally stable. Multiple poles at the same location on unit circle make the system unstable.

Two stability tests can be applied directly to the characteristic equation without solving for the roots.

→ Jury Stability test

→ Routh stability coupled with bi-linear transformation.

Jury Stability Test

Assume that the characteristic equation is as follows,

$$P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$$

Where $a_0 > 0$

Jury Table

Row	z^0	z^1	z^2	z^3	z^4	$\ldots$	z^n
1	a_n	a_{n-1}	$a_{n-2}\ldots$	$\ldots$	$\ldots$	a_0	
2	a_0	a_1	$a_2\ldots$	$\ldots$	$\ldots$	$\ldots$	a_n
3	b_{n-1}	b_{n-2}	$\ldots$	$\ldots$	b_0		
4	b_0	b_1	$\ldots$	$\ldots$	$\ldots$	b_{n-1}	
5	c_{n-2}	$c_{n-3}\ldots$	c_0				
6	c_0	$c_1\ldots$	$\ldots$	c_{n-2}			
.	....						
.	....						
.	....						
$2n-3$	q_2	q_1	q_0				

where,

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix}$$

$$k = 0, 1, 2, 3, ..., n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix}$$

$$k = 0, 1, 2, 3, ..., n-2$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix}$$

This system will be stable if:

1. $|a_n| \quad < a_0$
2. $P(z)|_{z=1} \quad > 0$
3. $P(z)|_{z=-1} \quad > 0$ for n even and $P(z)|_{z=-1} < 0$ for n odd
4.

$$\begin{array}{ll} |b_{n-1}| & > |b_0| \\ |c_{n-2}| & > |c_0| \\ . & .... \\ . & .... \\ |q_2| & > |q_0| \end{array}$$

Example : The characteristic equation is

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$$

Thus, $a_0 = 1 \qquad a_1 = -1.2 \qquad a_2 = 0.07 \qquad a_3 = 0.3 \qquad a_4 = -0.08$

We will now check the stability conditions.

1. $|a_n| = |a_4| = 0.08 < a_0 = 1 \Rightarrow$ First condition is satisfied.

2. $P(1) = 1 - 1.2 + 0.07 + 0.3 - 0.08 = 0.09 > 0 \Rightarrow$ Second condition is satisfied.

3. $P(-1) = 1 + 1.2 + 0.07 - 0.3 - 0.08 = 1.89 > 0 \Rightarrow$ Third condition is satisfied.

Next we will construct the Jury Table.

Jury Table

$$b_3 = \begin{vmatrix} a_n & a_0 \\ a_0 & a_n \end{vmatrix} = \quad 0.0064 - 1 = -0.9936$$

$$b_2 = \begin{vmatrix} a_n & a_1 \\ a_0 & a_3 \end{vmatrix} = \quad -0.08 \times 0.3 + 1.2 = 1.176$$

Rest of the elements are also calculated in a similar fashion. The elements are b_1=-0.0756

$b_0 = -0.204 \quad c_2 = 0.946 \quad c_1 = -1.184 \quad c_0 = 0.315$. One can see

$|b_3| = 0.9936 > |b_0| = 0.204$

$|c_2| = 0.946 > |c_0| = 0.315$

All criteria are satisfied. Thus the system is stable.

MODEL QUESTIONS

Module-2

Short Questions each carrying Two marks.

1. Determine the maximum phase lead of the compensator

$$D(s) = \frac{0.5s + 1}{0.5s + 1}$$

2. State the various effects and limitations of a lag compensator. Draw a representative sketch of a lag-lead compensator.

3. Derive Z transform of the following

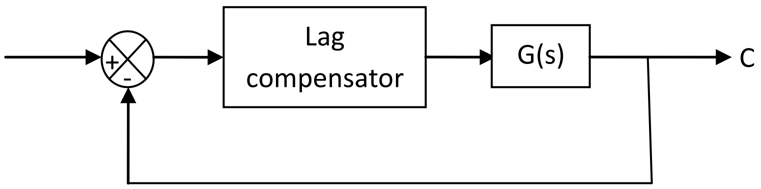
X(1)=2; X(4)=-3; X(7)=8

and all other samples are zero. Define the stability of discrete time function.

4. Find the inverse Z-transform if X(z)=Z.

The figures in the right-hand margin indicate marks.

5. The unity feedback system has the open loop plant $G(s) = \frac{1}{s(s+3)(s+6)}$



Design a lag compensation to meet the following specifications:

(i) Step response settling time < 5s.

(ii) Step response overshoot < 15%

(iii) Steady state error to a unit ramp input < 10%. **[10]**

6. A discrete time system is described by the difference equation

y(k+2) + 5y(k+1) + 6y (k) = U (k)

y(0) = y(1) =0; T=1sec

(i) Determine a state model in canonical form.

(ii) Find the output y(k) for the input u(k)=1 for ≥ 0. **[10]**

7. Use Jury's test to show that the two roots of the digital system $F(z)=Z^2+Z+0.25=0$ are inside the circle. **[3]**

8. **(a)**What is the principal effect of (i) lag, (ii) lead compensation on a root locus. **[3]**

(b) A type-1 unity feedback system has an open-loop transfer function

$$G(s) = \frac{K}{s(s+1)(0.2s+1)}$$

Design phase lag compensation for the system to achieve the following specifications:

Velocity error constant $K_v = 8$

Phase margin $\approx 40°$ **[13]**

9 . A discrete-time system is described by the difference equation

y(k+2)+5y(k+1) +6y(k) =u(k)

y(0)=y(1)=0; T=1 sec

(i) Determine a state model in canonical form

(ii) Find the state transition matrix

(iii) For input u(k)=1 for k ≥ 0, find the output y(k) **[5+5+6]**

10. Determine the z-transform of

(i) $F(s) = \frac{1}{(s+\alpha)^2}$

(ii) $F(s) = \frac{10(1-e^{-s})}{s(s+2)}$ **[8]**

11. Write short notes on **[4×7]**

(a) Feedback compensation

(b) stability analysis of sampled data control system

(c) R-C notch type a.c. lead network

(d) Hold circuits in sample data control

(e) Network compensation of a.c systems

(f) Z domain and s domain relationship

(g) Spectral factorisation

12. (a) Describe the effect of:

(i) Lag and

(ii) Lead compensation on a root locus **[4]**

(b) Design a suitable phase lag compensating network for a type-1 unity feedback system having an open-loop transfer function

$$G(s) = \frac{K}{s(0.1s+1)(0.2s+1)}$$

to meet the following specifications:

Velocity error constant $K_v = 30\text{sec}^{-1}$ and phase margin $\geq 40°$ **[12]**

13. Consider the system

$$\hat{X}(k+1) = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \hat{X}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$Y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} \hat{X}(k)$$

Find, if possible, a control sequence{u(0), u(1)} to drive the system from

$$\hat{X}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ to\ \hat{X}(2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$ **[8]**

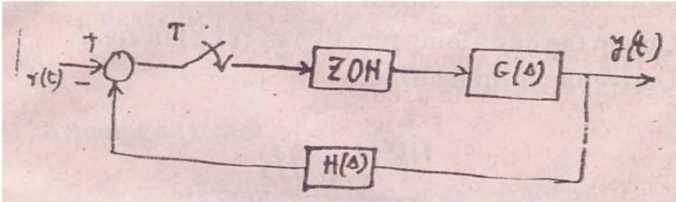
14. Find the inverse z-transform of

(i) $F(z) = \frac{10z}{(z-1)(z-2)}$

(ii) $F(z) = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$ **[4+4]**

15. Find Y(z) for the system in figure below, if r(t)=1(t), T=1 Sec,

$G(s) = \frac{1}{s+1}$, $H(s) = \frac{1}{s}$ **[7]**



16. Explain the relationship between s-plane & z-plane. **[3]**

17. How do you find out response between sampling instants? **[4]**

18. The open-loop transfer function of a servo mechanism is given by

$\boldsymbol{G(s)} = \frac{\mathbf{7}}{\boldsymbol{s}(\mathbf{1}+\mathbf{0.5}\boldsymbol{s})(\mathbf{1}+\mathbf{0.15}\boldsymbol{s})}$ **[15]**

Add series lag compensation to the servo mechanism to give a gain margin of $\geq$ 15dB and a phase margin $\geq 45^{\circ}$. Realise the compensator.

19.(a) Determine the state model of the system whose pulse transfer function given by

G(z)=$\frac{4z^3-12z^2+13z-7}{(z-1)^2(z-2)}$ **[8]**

(b) Find the z transform **[8]**

i)$\frac{3}{s^2+3s}$

ii) $4t^2$+10t+6

20. (a) Derive the transfer function of zero order hold circuit **[4×3]**

(b) State the specification in time domain and frequency domain used for the design of continuous time linear system.

(c) Explain how signal is reconstructed from the output of the sampler.

21. Find the z transform of the following transfer function. **[8]**

i) G(s)=$\frac{1}{(s+a)^2}$

ii) G(s)=$\frac{s+b}{(s+a)^2+w^2}$

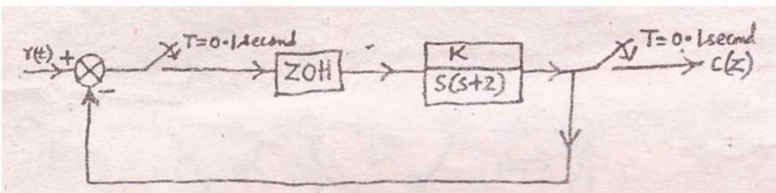
22.(a) A discrete time system is described by the difference equation **[7]**

Y(k+2)+5 y(k+1)+6 y(k) =u(k)

Y(0)=y(1)=0 u(k)=1 for k≥0

Find the output y(k)

(b) Find out the range of values of gain k for which the closed loop system shown in figure below remains stable. **[8]**



23. Prove that ZOH is a low pass filter. **[4]**

24. Explain what do you mean by aliasing in linear discrete data system. **[4]**

25.(a) Clearly explain how stability of sample data control system is assessed by Jury's stability test. **[7]**

(b) Check the stability of the linear discrete system having the characteristics equation:

$z^4 - 1.7z^3 + 1.04z^2 - 0.268z + 0.024 = 0$ **[8]**

26.(a) Determine the weighing sequence (impulse response) of linear discrete system described by

C(k)-αc(k-1)=r(k) **[7]**

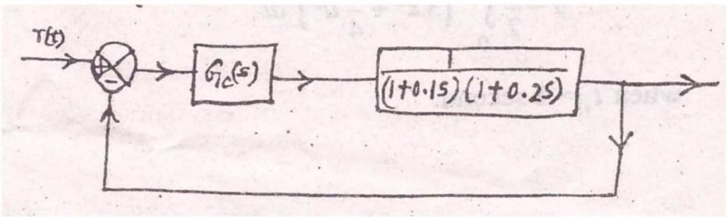
(b) With a neat circuit diagram, explain the principle of operation of sample and hold device. **[4]**

(c) Explain the significance of shanon's theorem in sampling process. **[4]**

27. A linear control system is to be compensated by a compensating network having

$G_C(S) = K_p + K_D s + \frac{K_i}{s}$

The system is shown in figure below



Find K_p,$K_D s$ and K_i so that the roots of the characteristics equation are placed at s= -50, -5 $\pm$ j5. **[9]**

28. A unity feedback system has an open loop transfer function of **[16]**

$G(s) = \frac{4}{s(2s+1)}$

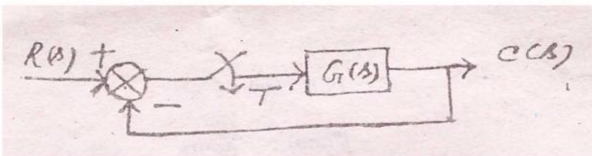
It is desired to obtain a phase margin of 40 degree without sacrificing the k_v of the system. Design a suitable lag network and compute the value of network components assuming any suitable impedance level.

29. (a) Find the z transform of the following: **[4+4]**

i)$y(t) = e^{-at} t^2$

ii)$G(s) = \frac{s+a}{(s+a)^2 + w^2}$

(b) For the system shown in fig below **[8]**



$G(z) = \frac{k(z+0.9)}{(z-1)(z-0.7)}$

Determine the range of k for stability.

MODULE-III

Introduction

A linear system designed to perform satisfactorily when excited by a standard test signal, will exhibit satisfactory behavior under any circumstances. Furthermore, the amplitude of the test signal is unimportant since any change in input signal amplitude results simply in change of response scale with no change in the basic response characteristics. The stability of nonlinear systems is determined solely by the location of the system poles & is independent entirely of whether or not the system is driven.

In contrast to the linear case, the response of nonlinear systems to a particular test signals is no guide to their behavior to other inputs, since the principle of superposition no longer holds. In fact, the nonlinear system response may be highly sensitive to the input amplitude. Here the stability dependent on the input & also the initial state. Further, the nonlinear systems may exhibit **limit cycle** which are self sustained oscillations of fixed frequency & amplitude.

Behaviour of Nonlinear Systems

A nonlinear system, when excited by a sinusoidal input, may generate several harmonics in addition to the fundamental corresponding to the input frequency. The amplitude of the fundamental is usually the largest, but the harmonics may be of significant amplitude in many situations.

Another peculiar characteristic exhibited by nonlinear systems is called jump phenomenon.

Jump Resonance

Consider the spring-mass-damper system as shown in Fig3.1(a). below.If the components are assumed to be linear, the system equation with a sinusoidal forcing function is given by

$$M\ddot{x} + f\dot{x} + Kx = F\cos wt \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3.1)$$

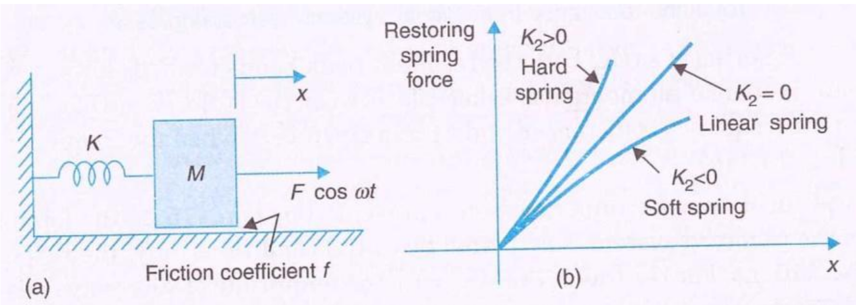


Fig.3.1 (a) A spring-mass-damper system (b)Spring Characteristics

The frequency response curve of this system is shown in Fig3.2.

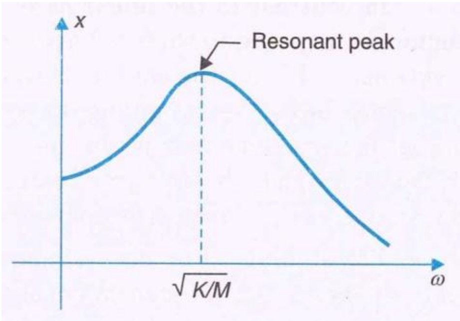


Fig. 3.2 Frequency response curve of spring-mass-damper system

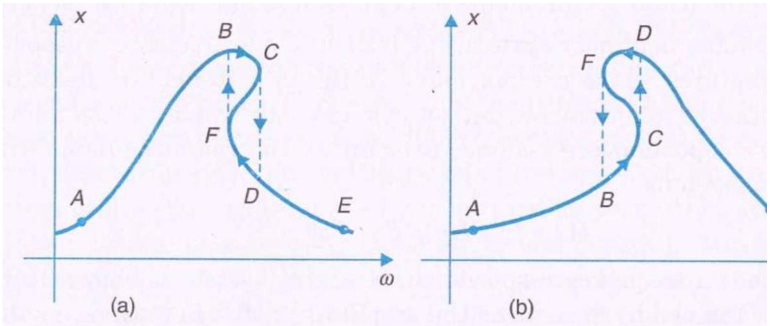


Fig. 3.3 (a) Jump resonance in nonlinear system(hard spring case);
(b) Jump resonance in nonlinear system(hard spring case).

Let us now assume that the restoring force of the spring is nonlinear,given by $K_1x + K_2x^3$.The nonlinear spring characteristic is shown in Fig.3.1(b). Now the system equation becomes

$$M\ddot{x} + f\dot{x} + K_1x + K_2x^3 = Fcos\ wt \dots\dots\dots\dots\dots\dots.(3.2)$$

The frequency response curve for the hard spring($K_2 > 0$) is shown in Fig3.3(a).

For a hard spring, as the input frequency is gradually increased from zero, the measured two response follows the curve through the A, B and C, but at C an increment in frequency results in discontinuous jump down to the point D, after which with further increase in frequency, the response curve follows through DE. If the frequency is now decreased, the response follows the curve EDF with a jump up to B from the point F and then the response curve moves towards A. This phenomenon which is peculiar to nonlinear systems is known as jump resonance. For a soft spring, jump phenomenon will happen as shown in fig. 3.3(b).

Methods of Analysis

Nonlinear systems are difficult to analyse and arriving at general conclusions are tedious. However, starting with the classical techniques for the solution of standard nonlinear differential equations, several techniques have been evolved which suit different types of analysis. It should be emphasised that very often the conclusions arrived at will be useful for the system under specified conditions and do not always lead to generalisations. The commonly used methods are listed below.

Linearization Techniques:

In reality all systems are nonlinear and linear systems are only approximations of the nonlinear systems. In some cases, the linearization yields useful information whereas in some other cases, linearised model has to be modified when the operating point moves from one to another. Many techniques like perturbation method, series approximation techniques, quasi-linearization techniques etc. are used for linearise a nonlinear system.

Phase Plane Analysis:

This method is applicable to second order linear or nonlinear systems for the study of the nature of phase trajectories near the equilibrium points. The system behaviour is qualitatively analysed along with design of system parameters so as to get the desired response from the system. The periodic oscillations in nonlinear systems called limit cycle can be identified with this method which helps in investigating the stability of the system.

Describing Function Analysis:

This method is based on the principle of harmonic linearization in which for certain class of nonlinear systems with low pass characteristic. This method is useful for the study of existence of limit cycles and determination of the amplitude, frequency and stability of these limit cycles. Accuracy is better for higher order systems as they have better low pass characteristic.

Classification of Nonlinearities:

The nonlinearities are classified into

i) Inherent nonlinearities and

ii) Intentional nonlinearities.

The nonlinearities which are present in the components used in system due to the inherent imperfections or properties of the system are known as inherent nonlinearities. Examples are saturation in magnetic circuits, dead zone, back lash in gears etc. However in some cases introduction of nonlinearity may improve the performance of the system, make the system more economical consuming less space and more reliable than the linear system designed to achieve the same objective. Such nonlinearities introduced intentionally to improve the system performance are known as intentional nonlinearities. Examples are different types of relays which are very frequently used to perform various tasks.

Common Physical Non Linearities:

The common examples of physical nonlinearities are saturation, dead zone, coulomb friction, stiction, backlash, different types of springs, different types of relays etc.

Saturation: This is the most common of all nonlinearities. All practical systems, when driven by sufficiently large signals, exhibit the phenomenon of saturation due to limitations of physical capabilities of their components. Saturation is a common phenomenon in magnetic circuits and amplifiers.

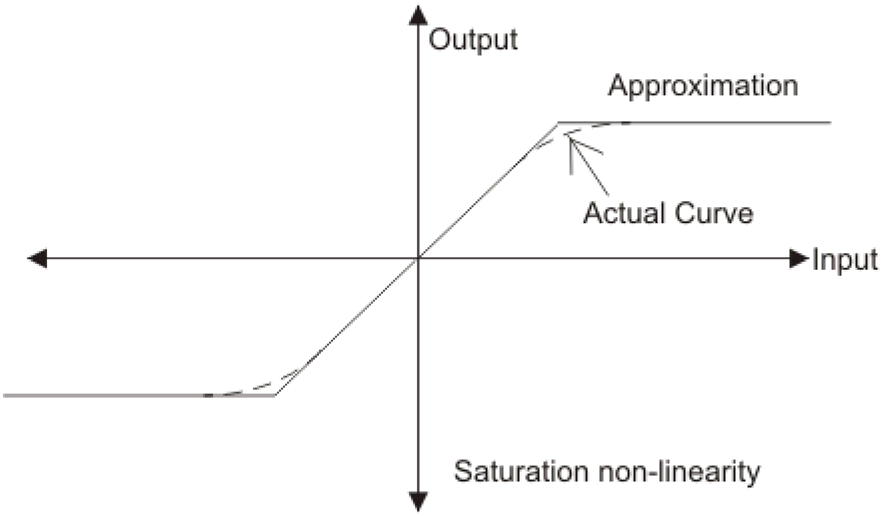


Fig. 3.4 Piecewise linear approximation of saturation nonlinearity

Friction: Retarding frictional forces exist whenever mechanical surfaces come in sliding contact. The predominant frictional force called the ***viscous friction*** is proportional to the relative velocity of sliding surfaces. Vicous friction is thus linear in nature. In addition to the viscous friction, there exist two nonlinear frictions. One is the ***coulomb friction*** which is constant retarding force & the other is the ***stiction*** which is the force required to initiate motion.The force of stiction is always greater than that of coulomb friction since due to interlocking of surface irregularities,more force is require to move an object from rest than to maintain it in motion.

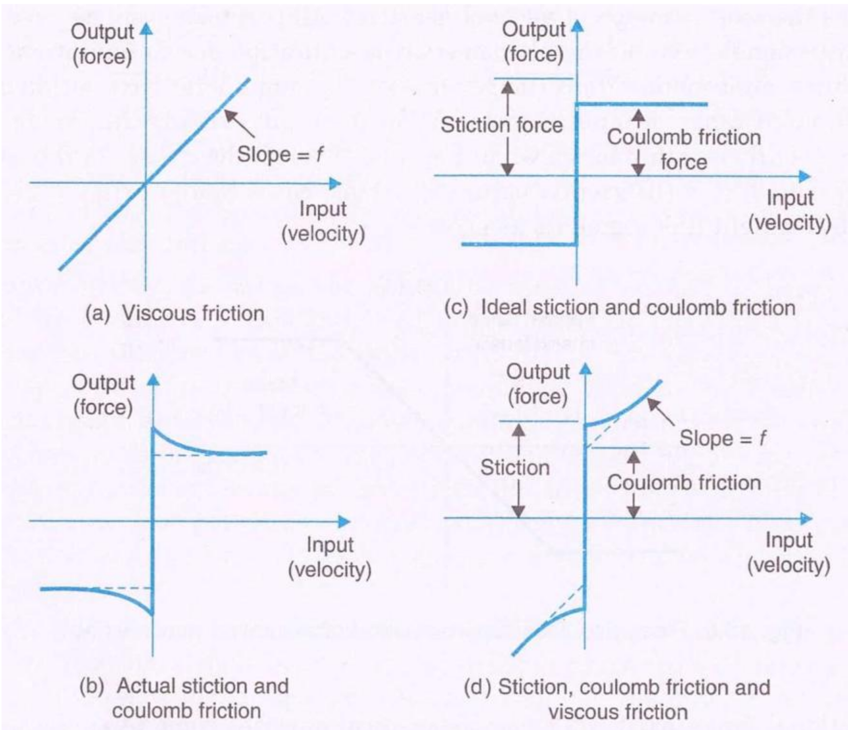


Fig. 3.5 Characteristics of various types of friction

Dead zone: Some systems do not respond to very small input signals. For a particular range of input, the output is zero. This is called dead zone existing in a system. The input-output curve is shown in figure.

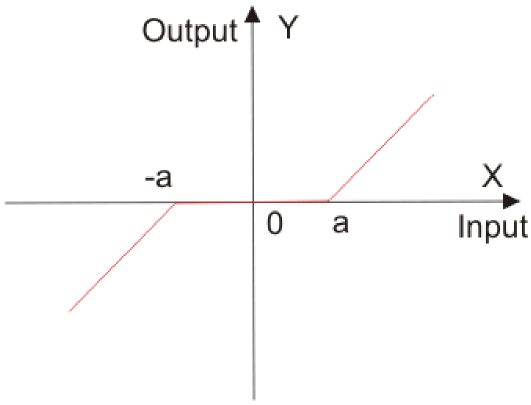


Fig. 3.6 Dead-zone nonlinearity

Backlash: Another important nonlinearity commonly occurring in physical systems is hysteresis in mechanical transmission such as gear trains and linkages. This nonlinearity is somewhat different from magnetic hysteresis and is commonly referred to as backlash. In servo systems, the gear backlash may cause sustained oscillations or chattering phenomenon and the system may even turn unstable for large backlash.

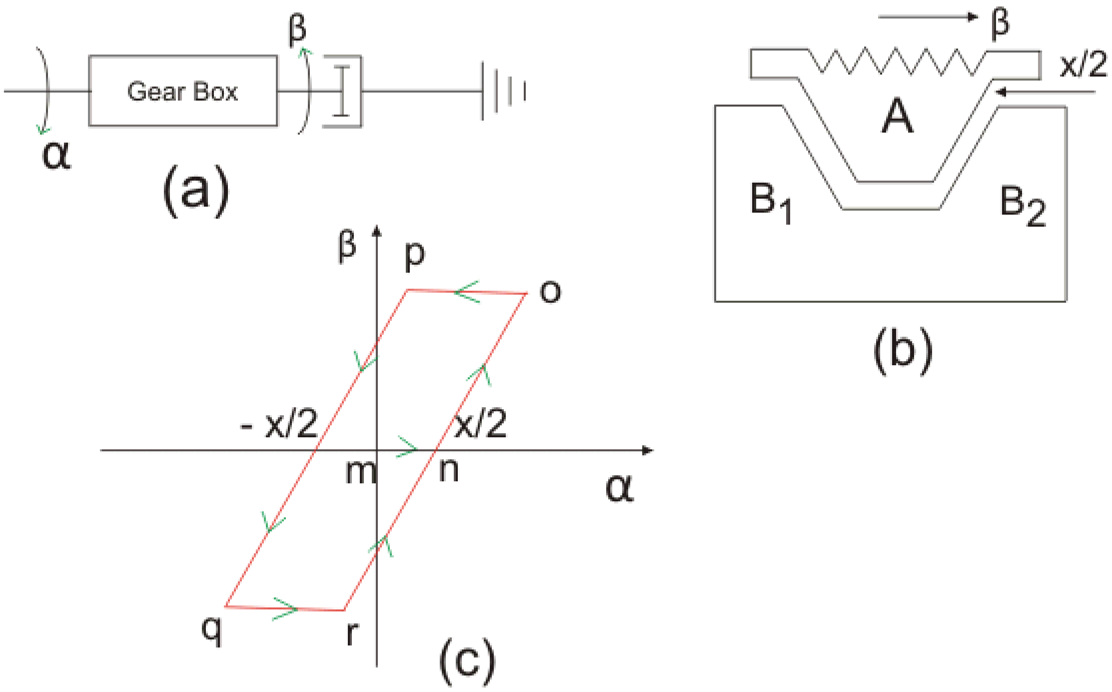


Figure 3.7: (a) gear box having backlash (b) the teeth A of the driven gear located midway between the teeth B_1, B_2 of the driven gear(c) gives the relationship between input and output motions.

As the teeth A is driven clockwise from this position, no output motion takes place until the tooth A makes contact with the tooth B_1 of the driven gear after travelling a distance x/2. This output motion corresponds to the segment mn of fig3.7 (c). After the contact is made the driven gear rotates counter clockwise through the same angle as the drive gear, if the gear ratio is assumed to be unity. This is illustrated by the line segment no. As the input motion is reversed, the contact between the teeth A and B_1 is lost and the driven gear immediately becomes stationary based on the assumption that the load is friction controlled with negligible inertia.

The output motion therefore causes till tooth A has travelled a distance x in the reverse direction as shown in fig3.7 (c) by the segment op. After the tooth A establishes contact with the tooth B_2, the driven gear now mores in clockwise direction as shown by segment pq. As the input motion is reversed the direction gear is again at standstill for the segment qr and then follows the drive gear along rn.

Relay: A relay is a nonlinear power amplifier which can provide large power amplification inexpensively and is therefore deliberately introduced in control systems. A relay controlled system can be switched abruptly between several discrete states which are usually off, full forward and full reverse. Relay controlled systems find wide applications in the control field. The characteristic of an ideal relay is as shown in figure. In practice a relay has a definite amount of dead zone as shown. This dead zone is caused by the facts that relay coil requires a finite amount of current to actuate the relay. Further, since a larger coil current is needed to close the relay than the current at which the relay drops out, the characteristic always exhibits hysteresis.

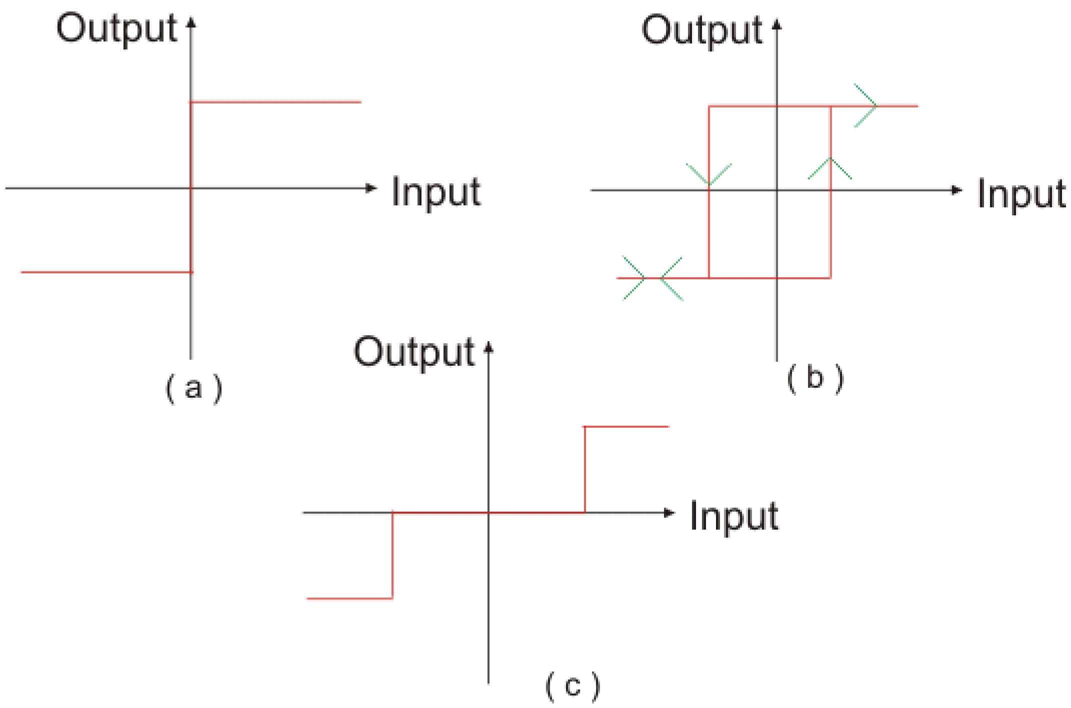


Figure3.8: Relay Non Linearity (a) ON/OFF (b) ON/OFF with Hysteresis (c) ON/OFF with Dead Zone

Multivariable Nonlinearity: Some nonlinearities such as the torque-speed characteristics of a servomotor, transistor characteristics etc., are functions of more than one variable. Such nonlinearities are called multivariable nonlinearities.

Phase Plane Analysis

Introduction

Phase plane analysis is one of the earliest techniques developed for the study of second order nonlinear system. It may be noted that in the state space formulation, the state variables chosen are usually the output and its derivatives. The phase plane is thus a state plane where the two state variables x_1 and x_2 are analysed which may be the output variable y and its derivative $\dot{y}$. The method was first introduced by Poincare, a French mathematician. The method is used for obtaining graphically a solution of the following two simultaneous equations of an autonomous system.

$$\dot{x_1} = f_1(x_1, x_2)$$
$$\dot{x_2} = f_2(x_1, x_2)$$

The $\dot{x_1} = f_1(x_1, x_2)$ & $\dot{x_2} = f_2(x_1, x_2)$ are either linear or nonlinear functions of the state variables x_1 and x_2 respectively. The state plane with coordinate axes x1 and x2 is called the ***phase plane***. In many cases, particularly in the phase variable representation of systems, take the form

$$\dot{x_1} = x_2$$
$$\dot{x_2} = f_2(x_1, x_2)$$

The curve described by the state point (x_1, x_2) in the phase plane with time as running parameter is called ***phase trajectory***.The plot of the state trajectories or phase trajectories of above said equation thus gives an idea of the solution of the state as time t evolves without explicitly solving for the state. The phase plane analysis is particularly suited to second order nonlinear systems with no input or constant inputs. It can be extended to cover other inputs as well such as ramp inputs, pulse inputs and impulse inputs.

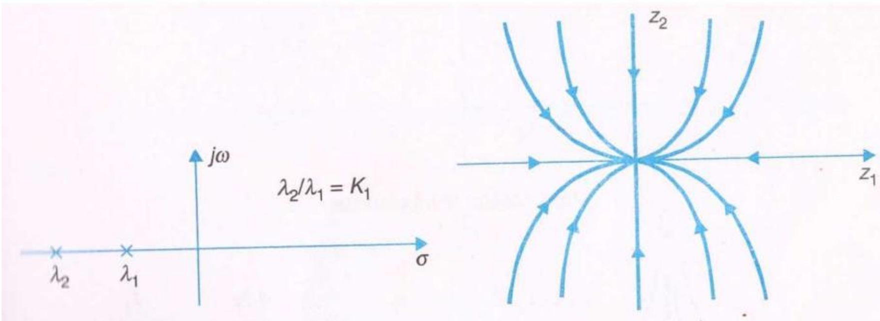
Phase Portraits

From the fundamental theorem of uniqueness of solutions of the state equations or differential equations, it can be seen that the solution of the state equation starting from an initial state in the state space is unique. This will be true if $f_1(x_1, x_2)$and $f_2(x_1, x_2)$ are analytic. For such a system, consider the points in the state space at which the derivatives of all the state variables are zero. These points are called ***singular points***. These are in fact ***equilibrium points*** of the system. If the system is placed at such a point, it will continue to lie there if left undisturbed. A family of phase trajectories starting from different initial states is called a ***phase portrait***. As time t increases, the phase portrait graphically shows how the system moves in the entire state plane from the initial states in the different regions. Since the solutions from each of the initial conditions are unique, the phase trajectories do not cross one another. If the system has nonlinear elements which are piece-wise linear, the complete state space can be divided into different regions and phase plane trajectories constructed for each of the regions separately.

Analysis & Classification of Singular Points

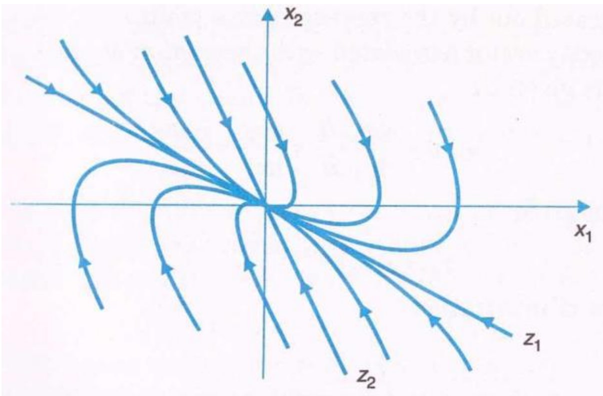
Nodal Point: Consider eigen values are real, distinct and negative as shown in figure3.9 (a). For this case the equation of the phase trajectory follows as $z_2 = c(z_1)^{\lambda_2/\lambda_1}$ Where c is an integration constant . The trajectories become a set of parabola as shown in figure 3.9(b) and the equilibrium point is called a node. In the original system of coordinates, these trajectories appear to be skewed as shown in figure 3.9(c).

If the eigen values are both positive, the nature of the trajectories does not change, except that the trajectories diverge out from the equilibrium point as both $z_1(t)$ and $z_2(t)$ are increasing exponentially. The phase trajectories in the x_1-x_2 plane are as shown in figure3.9 (d). This type of singularity is identified as a node, but it is an unstable node as the trajectories diverge from the equilibrium point.

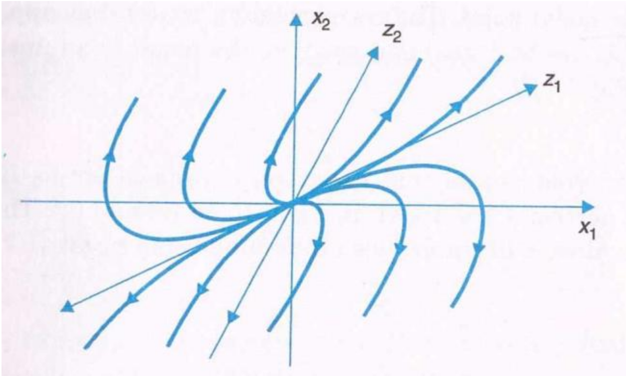


(a) Location of eigenvalues

(b) Stable node in (z_1, z_2)-plane



(c) Stable node in (X_1,X_2)-plane



(d) Unstable node in (X_1,X_2)-plane

Fig. 3.9

Saddle Point: Both eigen values are real,equal & negative of each other.The corresponding phase portraits are shown in Fig 3.10. The origin in this case a saddle point which is always unstable, one eigen value being positive.

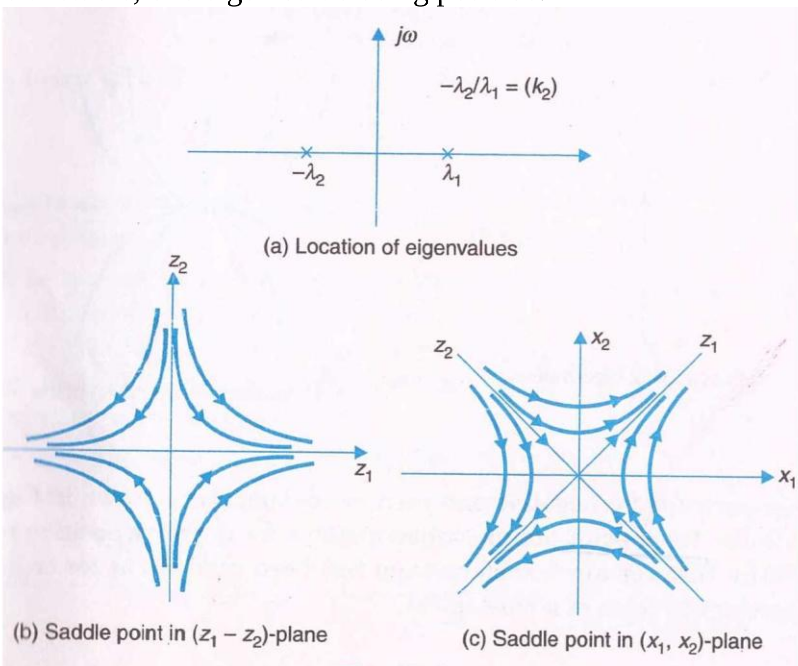


Fig 3.10

Focus Point: Consider a system with complex conjugate eigen values. A plot for negative values of real part is a family of equiangular spirals. Certain transformation has been carried out for (x_1, x_2) to (y_1, y_2) to present the trajectory in form of a true spiral. The origin which is a singular point in this case is called a stable focus. When the eigen values are complex conjugate with positive real parts, the phase portrait consists of expanding spirals as shown in figure and the singular point is an unstable focus. When transformed into the x1-x2 plane, the phase portrait in the above two cases is essentially spiralling in nature, except that the spirals are now somewhat twisted in shape.

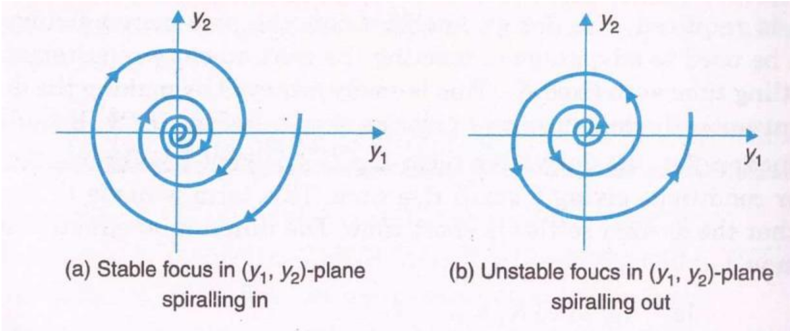


Fig 3.11

Centre or Vortex Point:

Consider now the case of complex conjugate eigen values with zero real parts.

ie., $\lambda_1, \lambda_2 = \pm j\omega$

$$\frac{dy_2}{dy_1} = \frac{jwy_1}{-jwy_2} = \frac{-y_1}{y_2} \quad \text{for which} \quad y_1 dy_1 + y_2 dy_2 = 0$$

Integrating the above equation, we get ${y_1}^2 + {y_2}^2 = R^2$ which is an equation to a circle of radius R. The radius R can be evaluated from the initial conditions. The trajectories are thus concentric circles in y1-y2 plane and ellipses in the x1-x2 plane as shown in figure. Such a singular points, around which the state trajectories are concentric circles or ellipses, are called a *centre* or *vortex*.

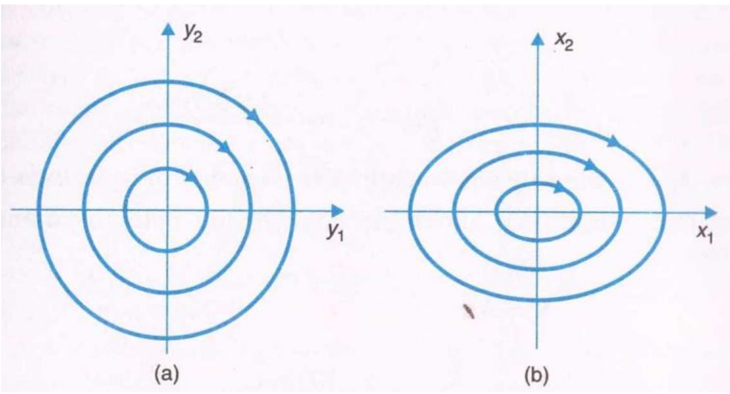


Fig.3.12 (a) Centre in (y_1,y_2)-plane (b) Centre in (X_1,X_2)-plane

Construction of Phase Trajectories:

Consider the homogenous second order system with differential equations

$$M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx = 0$$

$$\ddot{x} + 2\xi w_n \dot{x} + {w_n}^2 x = 0$$

where ζ and ωn are the damping factor and undamped natural frequency of the system.Defining the state variables as x = x_1 and $\dot{x} = x_2$, we get the state equation in the state variable form as

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -{w_n}^2 x_1 - 2\xi w_n x_2$$

These equations may then be solved for phase variables x_1 and x_2. The time response plots of x_1, x_2 for various values of damping with initial conditions can be plotted. When the differential equations describing the dynamics of the system are nonlinear, it is in general not possible to obtain a closed form solution of x_1, x_2. For example, if the spring force is nonlinear say ($k_1x + k_2x^3$) the state equation takes the form

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -\frac{k_1}{M}x_1 - \frac{f}{M}x_2 - \frac{k_2}{M}{x_1}^3$$

Solving these equations by integration is no more an easy task. In such situations, a graphical method known as the phase-plane method is found to be very helpful. The coordinate plane with axes that correspond to the dependent variable x_1 and x_2 is called phase-plane. The curve described by the state point (x_1,x_2) in the phase-plane with respect to time is called a phase trajectory. A phase trajectory can be easily constructed by graphical techniques.

Isoclines Method:

Let the state equations for a nonlinear system be in the form

$$\dot{x_1} = f_1(x_1, x_2)$$
$$\dot{x_2} = f_2(x_1, x_2)$$

When both f1(x1,x2) and f2(x1,x2) are analytic.
From the above equation, the slope of the trajectory is given by

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = M$$

Therefore, the locus of constant slope of the trajectory is given by $f_2(x_1,x_2) = Mf_1(x_1,x_2)$

The above equation gives the equation to the family of isoclines. For different values of M, the slope of the trajectory, different isoclines can be drawn in the phase plane. Knowing the value of M on a given isoclines, it is easy to draw line segments on each of these isoclines.

Consider a simple linear system with state equations

$$\dot{x_1} = x_2$$
$$\dot{x_2} = x_2 - x_1$$

Dividing the above equations we get the slope of the state trajectory in the x1-x2 plane as

$$\frac{dx_2}{dx_1} = \frac{x_2 - x_1}{x_2} = M$$

For a constant value of this slope say M, we get a set of equations

$$x_2 = \frac{-1}{M+1} x_1$$

which is a straight line in the x_1-x_2 plane. We can draw different lines in the x_1-x_2 plane for different values of M; called isoclines. If draw sufficiently large number of isoclines to cover the complete state space as shown, we can see how the state trajectories are moving in the state plane. Different trajectories can be drawn from different initial conditions. A large number of such trajectories together form a phase portrait. A few typical trajectories are shown in figure3.13 given below.

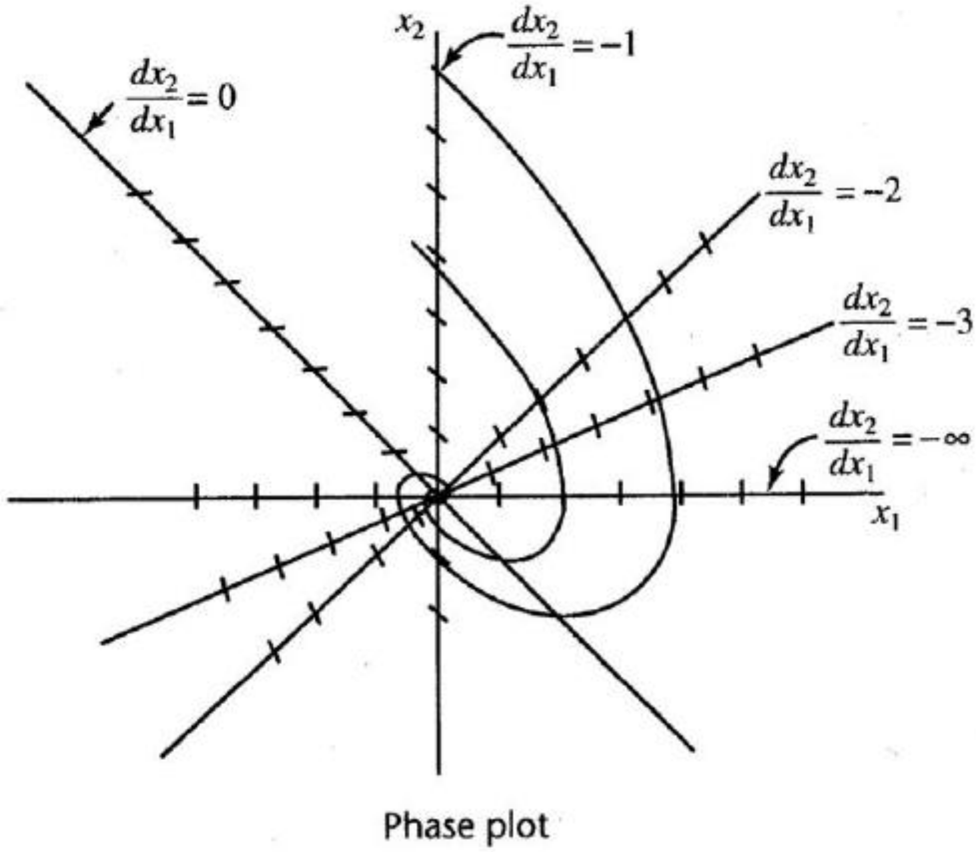


Fig. 3.13

The Procedure for construction of the phase trajectories can be summarised as below:

1.For the given nonlinear differential equation, define the state variables as x1 and x2 and obtain the state equations as

$$\dot{x_1} = x_2$$
$$\dot{x_2} = f_2(x_1, x_2)$$

2.Determine the equation to the isoclines as

$$\frac{dx_2}{dx_1} = \frac{f(x_1, x_2)}{x_2} = M$$

3. For typical values of M, draw a large number of isoclines in x1-x2 plane

4. On each of the isoclines, draw small line segments with a slope M.
5. From an initial condition point, draw a trajectory following the line segments With slopes M on each of the isoclines.

Delta Method:

The delta method of constructing phase trajectories is applied to systems of the form

$$\ddot{x} + f(x, \dot{x}, t) = 0$$

Where $f(x, \dot{x}, t)$ may be linear or nonlinear and may even be time varying but must be continuous and single valued.

With the help of this method, phase trajectory for any system with step or ramp or any time varying input can be conveniently drawn. The method results in considerable time saving when a single or a few phase trajectories are required rather than a complete phase portrait.

While applying the delta method, the above equation is first converted to the form

$$\ddot{x} + w_n[x + \delta(x, \dot{x}, t)] = 0$$

In general $\delta(x, \dot{x}, t)$ depends upon the variables $x, \dot{x}$ *and* t but for short intervals the changes in these variables are negligible. Thus over a short interval, we have

$\ddot{x} + w_n[x + \delta] = 0$, where δ is a constant.

Let us choose the state variables as $\dot{x_1} = x$, $x_2 = \dot{x}/w_n$, giving the state equations

$$\dot{x_1} = w_n x_2$$

$$\dot{x_2} = -w_n(x_1 + \delta)$$

Therefore, the slope equation over a short interval is given by

$$\frac{dx_2}{dx_1} = \frac{-x_1 + \delta}{x_2}$$

With δ known at any point P on the trajectory and assumed constant for a short interval, we can draw a short segment of the trajectory by using the trajectory slope dx_2/dx_1 given in the above equation. A simple geometrical construction given below can be used for this purpose.

1. From the initial point, calculate the value of δ.
2. Draw a short arc segment through the initial point with (-δ, 0) as centre, thereby determining a new point on the trajectory.
3. Repeat the process at the new point and continue.

Example : For the system described by the equation given below, construct the trajectory starting at the initial point (1, 0) using delta method.

$$\ddot{x} + \dot{x} + x^2 = 0$$

Let $= x_1$ and $\dot{x} = x_2$, then

$$\dot{x_1} = x_2$$
$$\dot{x_2} = x_2 - x_1^2$$

The above equation can be rearranged as

$$\dot{x_2} = -(x_1 + x_2 + x_1^2 - x_1)$$

So that

$$\delta = x_2 + x_1^2 - x_1$$

At initial point δ is calculated as δ = 0+1-1 = 0. Therefore, the initial arc is centered at point(0, 0). The mean value of the coordinates of the two ends of the arc is used to calculate the next value of δ and the procedure is continued. By constructing the small arcs in this way the arcs in this way the complete trajectory will be obtained as shown in figure3.14.

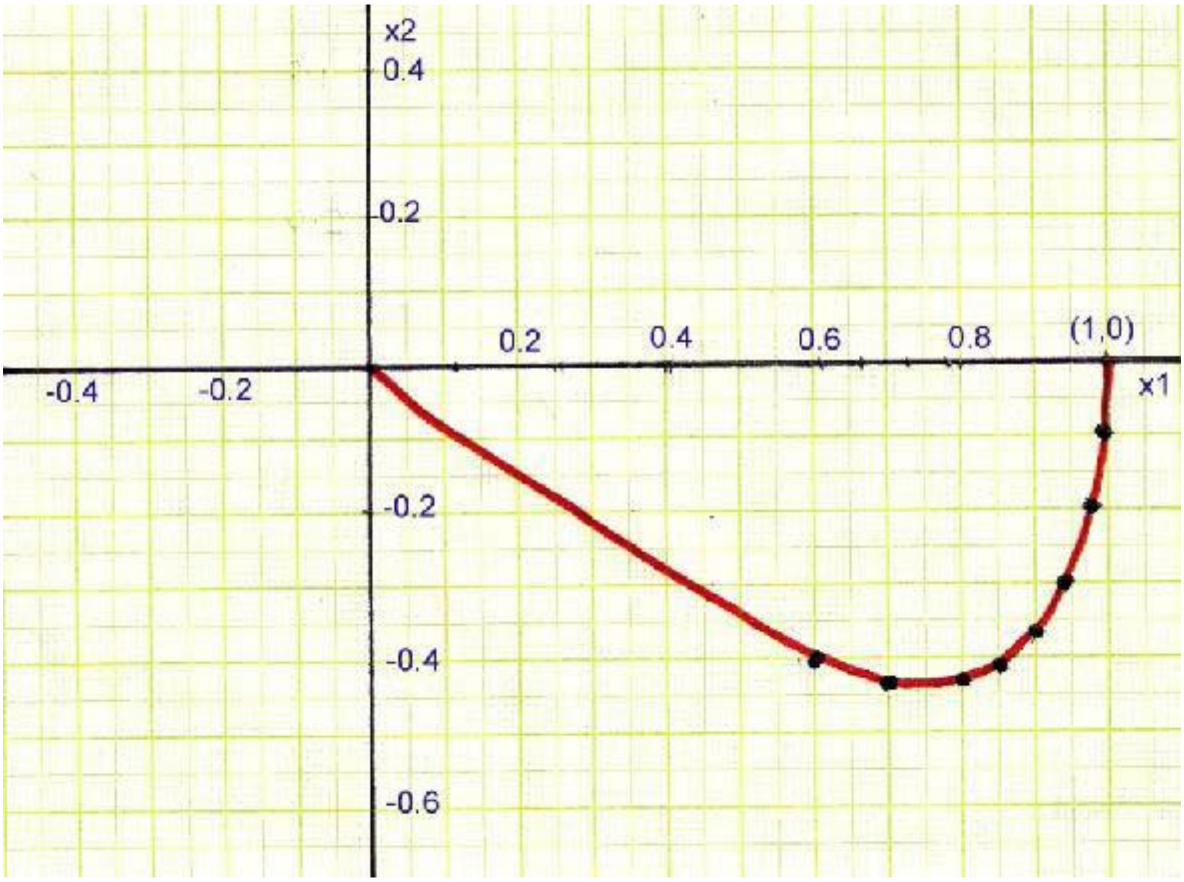


Fig.3.14

Limit Cycles:

Limit cycles have a distinct geometric configuration in the phase plane portrait, namely, that of an isolated closed path in the phase plane. A given system may have more than one limit cycle. A limit cycle represents a steady state oscillation, to which or from which all trajectories nearby will converge or diverge. In a nonlinear system, limit cycles describes the amplitude and period of a self sustained oscillation. It should be pointed out that not all closed curves in the phase plane are limit cycles. A phase-plane portrait of a conservative system, in which there is no damping to dissipate energy, is a continuous family of closed curves. Closed curves of this kind are not limit cycles because none of these curves are isolated from one another. Such trajectories always occur as a continuous family, so that there are closed curves in any neighborhoods of any particular closed curve. On the other hand, limit cycles are periodic motions exhibited only by nonlinear non conservative systems.

As an example, let us consider the well known Vander Pol's differential equation

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

which describes physical situations in many nonlinear systems.

In terms of the state variables $= x_1$ and $\dot{x} = x_2$, we obtained

$$\dot{x_1} = x_2$$

$$\dot{x_2} = \mu(1 - x_1^2)x_2 - x_1$$

The figure shows the phase trajectories of the system for μ > 0 and μ < 0. In case of μ > 0 we observe that for large values of $x_1(0)$, the system response is damped and the amplitude of $x_1(t)$ decreases till the system state enters the limit cycle as shown by the outer trajectory. On the other hand, if initially $x_1(0)$ is small, the damping is negative, and hence the amplitude of $x_1(t)$ increases till the system state enters the limit cycle as shown by the inner trajectory. When μ < 0, the trajectories moves in the opposite directions as shown in figure3.15.

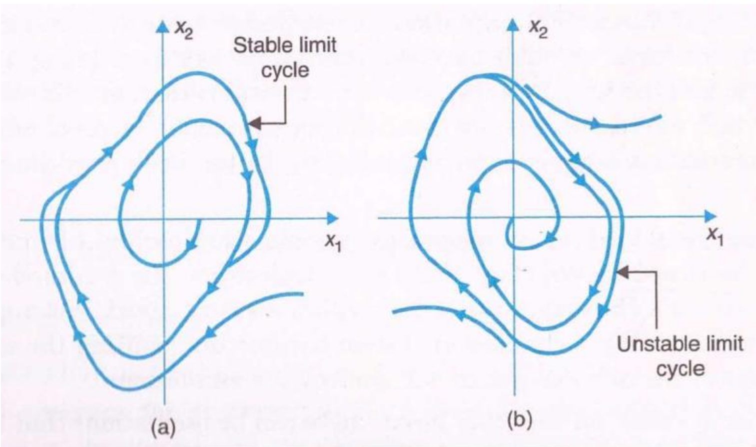


Fig.3.15 Limit cycle behavior of nonlinear system

A limit cycle is called stable if trajectories near the limit cycle, originating from outside or inside, converge to that limit cycle. In this case, the system exhibits a sustained oscillation with constant amplitude. This is shown in figure (i). The inside of the limit cycle is an unstable region in the sense that trajectories diverge to the limit cycle, and the outside is a stable region in the sense that trajectories converge to the limit cycle.
A limit cycle is called an unstable one if trajectories near it diverge from this limit cycle. In this case, an unstable region surrounds a stable region. If a trajectory starts within the stable region, it converges to a singular point within the limit cycle. If a trajectory starts in the unstable region, it diverges with time to infinity as shown in figure (ii). The inside of an unstable limit cycle is the stable region, and the outside the unstable region.

Describing Function Method of Non Linear Control System

Describing function method is used for finding out the stability of a non linear system. Of all the analytical methods developed over the years for non linear control systems, this method is generally agreed upon as being the most practically useful. This method is basically an approximate extension of frequency response methods including Nyquist stability criterion to non linear system.

The **describing function method** of a non linear system is defined to be the complex ratio of amplitudes and phase angle between fundamental harmonic components of output to input sinusoid. We can also called sinusoidal describing function. Mathematically,

$$N = \frac{Y_1}{X}\angle\phi_1$$

Where, N = describing function,

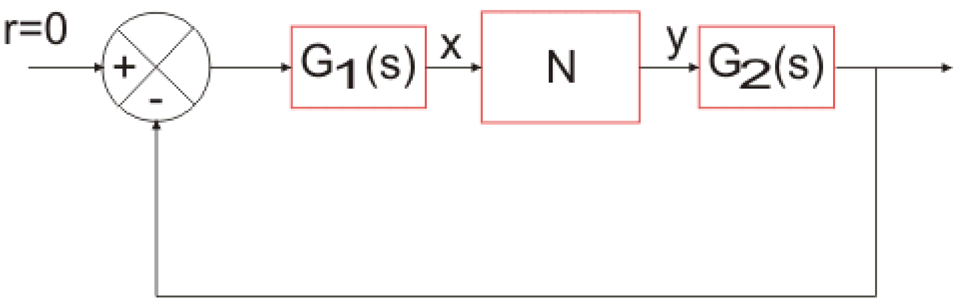
X = amplitude of input sinusoid,

Y = amplitude of fundamental harmonic component of output,

φ_1 = phase shift of the fundamental harmonic component of output.

Let us discuss the basic concept of describing function of non linear control system.

Let us consider the below block diagram of a non linear system, where $G_1(s)$ and $G_2(s)$ represent the linear element and N represent the non linear element.



Let us assume that input x to the non linear element is sinusoidal, i.e,

$$x = X \sin \omega t$$

For this input, the output y of the non linear element will be a non sinusoidal periodic function that may be expressed in terms of Fourier series as

$$y = Y_0 + A_1 \cos \omega t + B_1 \sin \omega t + A_2 \cos 2\omega t + B_2 \sin 2\omega t + \cdots$$

Most of non linearities are odd symmetrical or odd half wave symmetrical; the mean value Y_0 for all such case is zero and therefore output will be,

$$y = A_1 \cos \omega t + B_1 \sin \omega t + A_2 \cos 2\omega t + B_2 \sin 2\omega t + \cdots$$

As $G_1(s)$ $G_2(s)$ has low pass characteristics , it can be assumed to a good degree of approximation that all higher harmonics of y are filtered out in the process, and the input x to the nonlinear element N is mainly contributed by fundamental component of y i.e. first harmonics . So in the describing function analysis, we assume that only the fundamental harmonic component of the output. Since the higher harmonics in the output of a non linear system are often of smaller amplitude than the amplitude of fundamental harmonic component. Most control systems are low pass filters, with the result that the higher harmonics are very much attenuated compared with the fundamental harmonic component. Hence y_1 need only be considered.

$$y_1 = A_1 \cos \omega t + B_1 \sin \omega t$$

We can write $y_1(t)$ in the form ,

$$y_1(t) = A_1 \sin(\omega t + 90^\circ) + B_1 \sin \omega t = Y_1 \sin(\omega t + \phi_1)$$

Where by using phasor,

$$Y_1 \angle \phi_1 = B_1 + jA_1 = \sqrt{B_1^2 + A_1^2} \angle \tan^{-1}\left(\frac{A_1}{B_1}\right)$$

The coefficient A_1 and B_1 of the Fourier series are given by-

$$A_1 = \frac{1}{\pi} \int_0^{2\pi} y \cos \omega t d(\omega t)$$

$$B_1 = \frac{1}{\pi} \int_0^{2\pi} y \sin \omega t d(\omega t)$$

From definition of describing function we have,

$$N = \frac{Y_1}{X} \angle \phi_1 = \frac{\sqrt{A_1^2 + B_1^2}}{X} \angle \tan^{-1}\left(\frac{A_1}{B_1}\right)$$

Describing Function for Saturation Non Linearity

We have the characteristic curve for saturation as shown in the given figure3.16

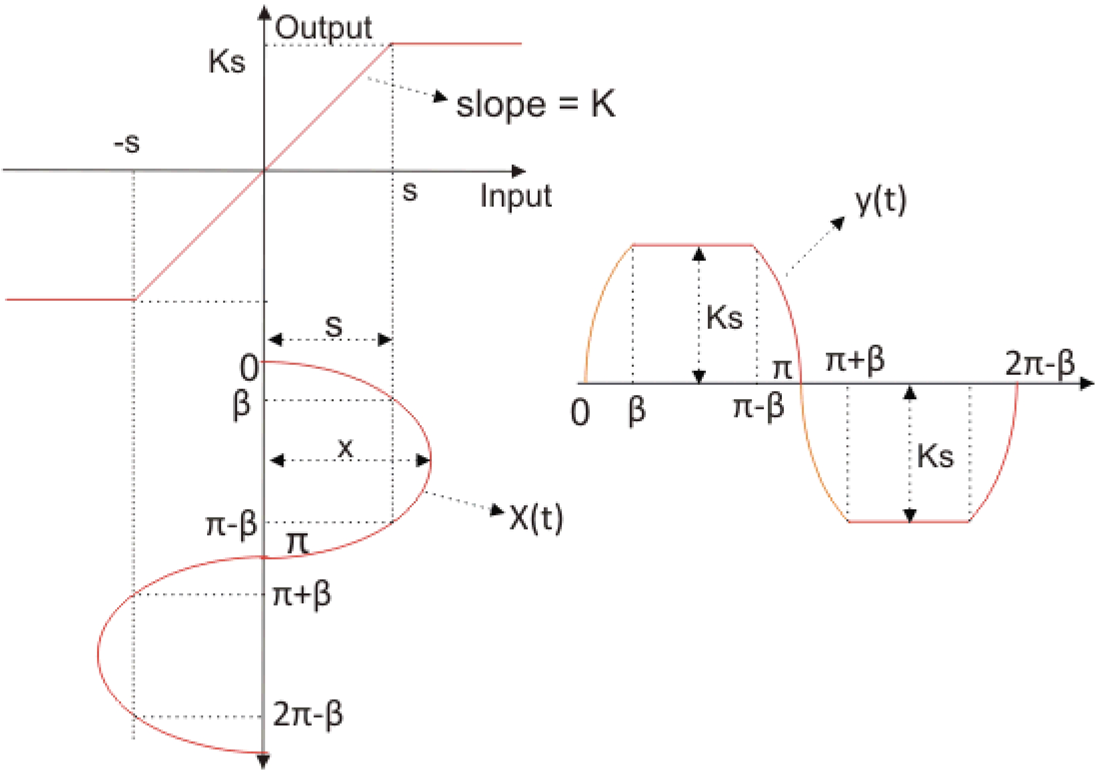


Fig. 3.16. Characteristic Curve for Saturation Non Linearity.

Let us take input function as

$$X(t) = X\sin(\omega t).$$

Now from the curve we can define the output as :

$$Y(t) = KX\sin\omega t \; for \; 0 \leq \omega t \leq \beta$$
$$Y(t) = Ks \; for \; \beta \leq \omega t \leq (\pi - \beta)$$
$$Y(t) = KX\sin\omega t \; for \; (\pi - \beta) \leq \omega t \leq \pi$$

Let us first calculate Fourier series constant A_1.

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos\omega t \; d(\omega t)$$

On substituting the value of the output in the above equation and integrating the function from 0 to 2π we have the value of the constant A1 as zero.

Similarly we can calculate the value of Fourier constant B_1 for the given output and the value of B_1 can be calculated as,

$$
\begin{aligned}
B_1 &= \frac{1}{\pi}\int_0^{2\pi} y(t)\sin\omega t\; d(\omega t) \\
&= \frac{4}{\pi}\int_0^{\pi/2} y(t)\sin\omega t\; d(\omega t) \\
&= \frac{1}{\pi}\left[\int_0^{\beta} KX\sin^2\omega t\; d(\omega t) + \int_{\beta}^{\pi/2} Ks\times\sin\omega t\; d(\omega t)\right] \\
&= \frac{4K}{\pi}\left[\frac{X\beta}{2} - \frac{X}{4}\sin 2\beta + s\cos\beta\right] \\
&= \frac{2KX}{\pi}\left[\beta + 2\frac{S}{X}\cos\beta - \sin\beta\cos\beta\right]
\end{aligned}
$$

$$
\begin{aligned}
&Now\ KX\sin\omega t = Ks\ when\ \omega t = \beta \\
&So,\ \sin = \frac{Ks}{KX} \\
&\Rightarrow \beta = \sin^{-1}\frac{s}{X} \\
&\therefore B_1 = \frac{2KX}{\pi}\left[\sin^{-1}\left(\frac{s}{X}\right) + 2\frac{S}{X}\cos\sin^{-1}\left(\frac{s}{X}\right) - \sin\sin^{-1}\left(\frac{s}{X}\right).\cos\sin^{-1}\left(\frac{s}{X}\right)\right] \\
&= \frac{2KX}{\pi}\left[\sin^{-1}\left(\frac{s}{X}\right) + \frac{s}{X}\sqrt{1-\left(\frac{s}{X}\right)^2}\right]
\end{aligned}
$$

The phase angle for the describing function can be calculated as

$$\angle\tan^{-1}\left(\frac{A_1}{B_1}\right) = \angle 0^\circ$$

Thus the describing function for saturation is

$$
\begin{aligned}
N &= \frac{B_1}{X}\angle 0^\circ \\
&= \frac{2K}{\pi}\left[\sin^{-1}\left(\frac{s}{X}\right) + \frac{s}{X}\sqrt{1-\left(\frac{s}{X}\right)^2}\right]\angle 0^\circ
\end{aligned}
$$

Describing Function for Ideal Relay

We have the characteristic curve for ideal relay as shown in the given figure 3.17.

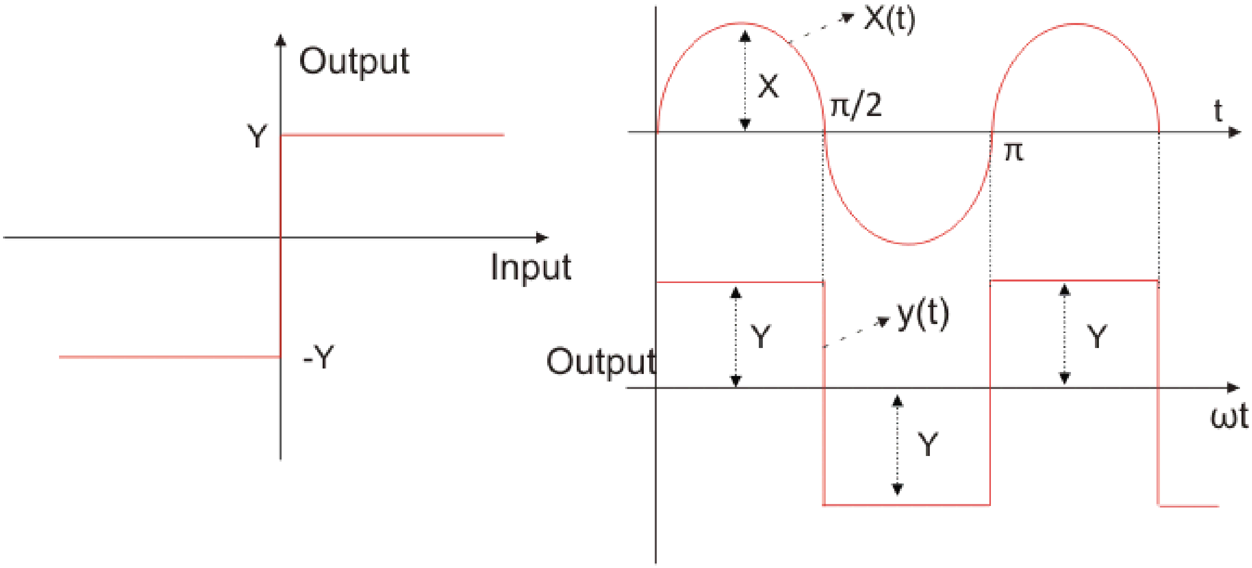


Fig. 3.17. Characteristic Curve for Ideal Relay Non Linearity.

Let us take input function as

$$X(t) = X\sin(\omega t).$$

Now from the curve we can define the output as

$$Y(t) = Y \;\; for \; 0 \le \omega t \le \pi$$
$$Y(t) = -Y \;\; for \; \pi \le \omega t \le 2\pi$$

The output periodic function has odd symmetry :

$$y(\omega t) = -y(-\omega t)$$

Let us first calculate Fourier series constant A_1.

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos\omega t \; d(\omega t)$$

On substituting the value of the output in the above equation and integrating the function from 0 to 2π we have the value of the constant A_1 as zero.

Similarly we can calculate the value of Fourier constant B_1 for the given output and the value of B_1 can be calculated as

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin\omega t \ d(\omega t)$$
$$= \frac{2}{\pi}\int_0^{\pi} y(t)\ \sin\omega t \ d(\omega t)$$

On substituting the value of the output in the above equation y(t) = Y we have the value of the constant B_1

$$B_1 = \frac{2Y}{\pi}\int_0^{\pi} y(t)\sin\omega t \ d(\omega t) = \frac{4Y}{\pi}$$

And the phase angle for the describing function can be calculated as

$$\angle \tan^{-1}\left(\frac{A_1}{B_1}\right) = \angle 0^\circ$$

Thus the describing function for an ideal relay is

$$N = \frac{Y_1}{X}\angle 0^\circ = \frac{4Y}{\pi X}\angle 0^\circ$$

Describing Function for Real Relay (Relay with Dead Zone)

We have the characteristic curve for real realy as shown in the given figure 3.18. If X is less than dead zone Δ, then the relay produces no output; the first harmonic component of Fourier series is of course zero and describing function is also zero. If X > &Delta, the relay produces the output.

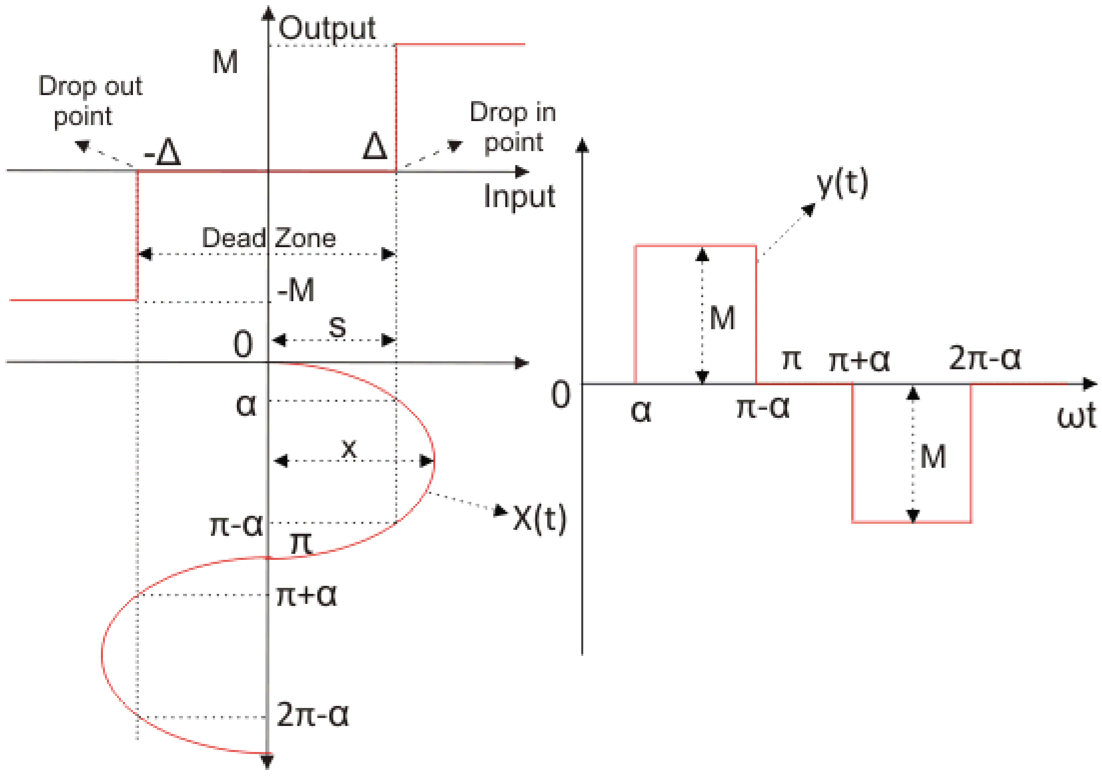


Fig. 3.18. Characteristic Curve for Real Relay Non Linearities.

Let us take input function as

$$X(t) = X\sin(\omega t).$$

Now from the curve we can define the output as

$$Y(t) = 0 \ for \ 0 \le \omega t \le \alpha$$
$$Y(t) = M \ for \ \alpha \le \omega t \le (\pi - \alpha)$$
$$Y(t) = 0 \ for \ (\pi - \alpha) \le \omega t \le \pi$$
$$Y(t) = -M \ for \ (\pi + \alpha) \le \omega t \le (2\pi - \alpha)$$
$$Y(t) = 0 \ for \ (2\pi - \alpha) \le \omega t \le 2\pi$$

$$Where, \ X\sin\alpha = \Delta$$
$$\Rightarrow \alpha = \sin^{-1}\left(\frac{\Delta}{X}\right)$$

The output periodic function has odd symmetry :

$$y(\omega t) = -y(-\omega t)$$

Let us first calculate Fourier series constant A_1.

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos\omega t \ d(\omega t)$$

On substituting the value of the output in the above equation and integrating the function from 0 to 2π we have the value of the constant A_1 as zero.

Similarly we can calculate the value of Fourier constant B for the given output and the value of B can be calculated as

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y \sin \omega t \ d(\omega t)$$

Due to the symmetry of y, the coefficient B_1 can be calculated as follows,

$$\begin{aligned} B_1 &= \frac{4}{\pi}\int_0^{\pi/2} y \sin \omega t \ d(\omega t) \\ &= \frac{4M}{\pi}\int_\alpha^{\pi/2} y \sin \omega t \ d(\omega t) \\ &= \frac{4M}{\pi}\cos \alpha \end{aligned}$$

Therefore, the describing function is

$$\begin{aligned} N &= \frac{4M}{\pi X}\cos \alpha \ \angle 0^\circ \\ &= \frac{4M}{\pi X}\sqrt{1 - \left(\frac{\Delta}{X}\right)^2} \ \angle 0^\circ \quad \left[Since \ \sin \alpha = \frac{\Delta}{X} \right] \end{aligned}$$

Describing Function for Backlash Non Linearity

We have the characteristic curve for backlash as shown in the given figure 3.19.

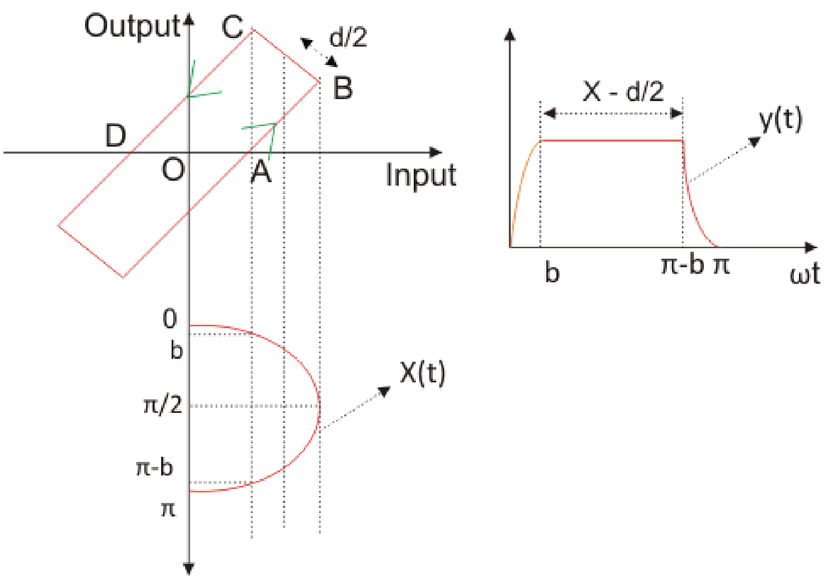


Fig. 3.19. Characteristic Curve of Backlash Non Linearity.

Let us take input function as

$$X(t) = X\sin(\omega t).$$

Now from the curve we can define the output as

$$Y(t) = K\left[X\sin\left(\omega t - \frac{d}{2}\right)\right] \quad for \;\; 0 \le \omega t \le \frac{\pi}{2}$$
$$Y(t) = K\left[X - \frac{d}{2}\right] \quad for \;\; \frac{\pi}{2} \le \omega t \le (\pi - b)$$
$$Y(t) = K\left[X\sin\left(\omega t + \frac{d}{2}\right)\right] \quad for \;\; (\pi - b) \le \omega t \le \pi$$

Let us first calculate Fourier series constant A_1.

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos\omega t \; d(\omega t)$$

On substituting the value of the output in the above equation and integrating the function from zero to 2π we have the value of the constant A_1 as

$$A_1 = \frac{4KX}{\pi}\left[\frac{(d/2)^2}{X^2} - \frac{(d/2)}{X}\right]$$

Similarly we can calculate the value of Fourier constant B for the given output and the value of B_1 can be calculated as

$$B_1 = \frac{1}{2\pi}\int_0^{2\pi} y(t)\sin\omega t \; d(\omega t)$$

On substituting the value of the output in the above equation and integrating the function from zero to pi we have the value of the constant B_1 as

$$B_1 = \frac{KX}{\pi}\left[\frac{\pi}{2} + b + \frac{d(X-d)}{X^2}\sqrt{\frac{2X}{d} - 1}\right]$$

We can easily calculate the describing function of backlash from below equation

$$N = \frac{\sqrt{A_1^2 + B_1^2}}{X}\angle\tan^{-1}\left(\frac{A_1}{B_1}\right)$$

Liapunov's Stability Analysis

Consider a dynamical system which satisfies

$\dot{x} = f(x, t)$; with initial condition $x(t_o) = x_o$; $x \in R_n$ ... ... (3.3)

We will assume that $f(x, t)$ satisfies the standard conditions for the existence and uniqueness of solutions. Such conditions are, for instance, that $f(x, t)$ is Lipschitz continuous with respect to x, uniformly in t, and piecewise continuous in t. A point $x^* \in R_n$ is an *equilibrium point* of equation (3.3) if $F(x^*, t) \equiv 0$.

Intuitively and somewhat crudely speaking, we say an equilibrium point is ***locally stable*** if all solutions which start near x^* (meaning that the initial conditions are in a neighborhood of x^* remain near x^* for all time.

The equilibrium point x^* is said to be ***locally Asymptotically stable*** if x^* is locally stable and, furthermore, all solutions starting near x^* tend towards x^* as $t \to \infty$.

We say somewhat crude because the time-varying nature of equation (3.3) introduces all kinds of additional subtleties. Nonetheless, it is intuitive that a pendulum has a locally stable equilibrium point when the pendulum is hanging straight down and an unstable equilibrium point when it is pointing straight up. If the pendulum is damped, the stable equilibrium point is locally asymptotically stable. By shifting the origin of the system, we may assume that the equilibrium point of interest occurs at $x^* = 0$. If multiple equilibrium points exist, we will need to study the stability of each by appropriately shifting the origin.

3.1 Stability in the sense of Lyapunov

The equilibrium point $x^* = 0$ of (3.3) is *stable (in the sense of Lyapunov)* at $t = t_0$ if for any $\epsilon > 0$ there exists a $\delta(t_0, \epsilon) > 0$ such that

$$\|x(t_o)\| < \delta \;\Rightarrow\; \|x(t)\| < \epsilon \;, \quad \forall t \geq t_o \;\ldots\ldots\ldots\ldots\ldots..(3.4)$$

Lyapunov stability is a very mild requirement on equilibrium points. In particular, it does not require that trajectories starting close to the origin tend to the origin asymptotically. Also, stability is defined at a time instant t_0. ***Uniform stability*** is a concept which guarantees that the equilibrium point is not losing stability. We insist that for a uniformly stable equilibrium point x^*, δ in the Definition 3.1 not be a function of t_0, so that equation (3.4) may hold for all t_0. Asymptotic stability is made precise in the following definition:

3.2 Asymptotic stability

An equilibrium point $x^* = 0$ of (3.3) is *asymptotically stable* at $t = t_0$ if
1. $x^* = 0$ is stable, and
2. $x^* = 0$ is locally attractive; i.e., there exists $\delta(t_0)$ such that

$$\|x(t_o)\| < \delta \;\Rightarrow\; \lim_{t\to\infty} x(t) = 0, \qquad \ldots\ldots\ldots\ldots\ldots..(3.5)$$

As in the previous definition, asymptotic stability is defined at t_0.

Uniform asymptotic stability requires:

1. $x^* = 0$ is uniformly stable, and
2. $x^* = 0$ is uniformly locally attractive; i.e., there exists δ independent of $t0$ for which equation (3.5) holds. Further, it is required that the convergence in equation (3.5) is uniform.

Finally, we say that an equilibrium point is *unstable* if it is not stable. This is less of a tautology than it sounds and the reader should be sure he or she can negate the definition of stability in the sense of Lyapunov to get a definition of instability. In robotics, we are almost always interested in uniformly asymptotically stable equilibria. If we wish to move the robot to a point, we would like to actually converge to that point, not merely remain nearby. Figure below illustrates the difference between stability in the sense of Lyapunov and asymptotic stability.

Definitions 3.1 and 3.2 are *local* definitions; they describe the behavior of a system near an equilibrium point. We say an equilibrium point x^* is *globally* stable if it is stable for all initial conditions $x_0 \in R_n$. Global stability is very desirable, but in many applications it can be difficult to achieve. We will concentrate on local stability theorems and indicate where it is possible to extend the results to the global case. Notions of uniformity are only important for time-varying systems. Thus, for time-invariant systems, stability implies uniform stability and asymptotic stability implies uniform asymptotic stability.

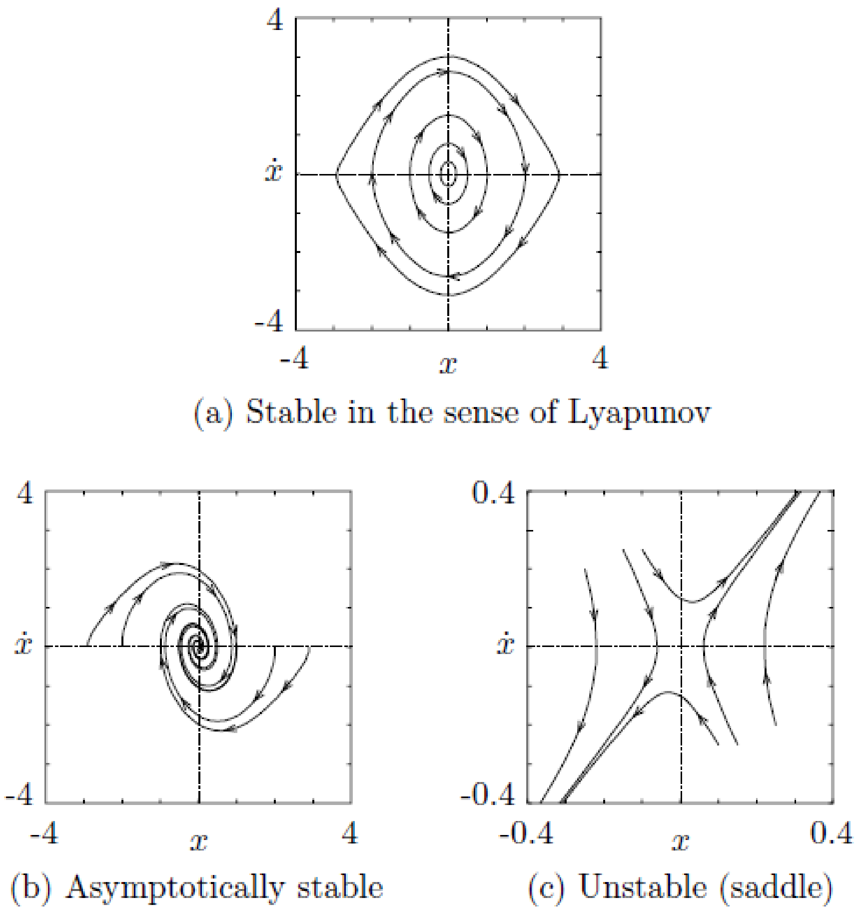


Figure:3.20 Phase portraits for stable and unstable equilibrium points.

Basic theorem of Lyapunov

Let V (x, t) be a non-negative function with derivative $\dot{V}$ along the trajectories of the system.

1. *If V (x, t) is locally positive definite and $\dot{V}(x, t) \le 0$ locally in x and for all t, then the origin of the system is locally stable (in the sense of Lyapunov).*

2. *If V (x, t) is locally positive definite and decrescent, and $\dot{V}(x, t) \le 0$ locally in x and for all t, then the origin of the system is uniformly locally stable (in the sense of Lyapunov).*

3. *If V (x, t) is locally positive definite and decrescent, and $-\dot{V}(x, t)$ is locally positive definite, then the origin of the system is uniformly locally asymptotically stable.*

4. *If V (x, t) is positive definite and decrescent, and $-\dot{V}(x, t)$ is positive definite, then the origin of the system is globally uniformly asymptotically stable.*

Theorm-1

Consider the system

$$\dot{x} = f(x);\ f(0) = 0$$

Suppose there exists a scalar function v(x) which for some real number $\in > 0$ satisfies the following properties for all x in the region $\|x(t)\| < \epsilon$

(a) V(x)>0; $x \neq 0$ that is v(x) is positive definite scalar function.
(b) V (0) = 0
(c)V(x) has continuous partial derivatives with respect to all component of x
(d) $\frac{dv}{dt} \le 0$ (i.e dv/dt is negative semi definite scalar function)
Then the system is stable at the origin

Theorem-2

If the property of (d) of theorem-1 is replaced with (d) $\frac{dv}{dt} < 0, x \neq 0$ (i.e dv/dt is negative definite scalar function),then the system is asymptotically stable.
It is intuitively obvious since continuous v function>0 except at x=0, satisfies the condition dv/dt <0, we except that x will eventually approach the origin .We shall avoid the rigorous of this theorem.

Theorem-3

If all the conditions of theorem-2 hold and in addition.

$$V(x) \to \infty \quad as \quad \|x\| \to \infty$$

Then the system is asymptotically stable in-the-large at the origin.

Instability

It may be noted that instability in a nonlinear system can be established by direct recourse to the instability theorem of the direct method .The basic instability theorem is presented below:

Theorem-4
Consider a system

$$\dot{x} = f(x);\ f(0) = 0$$

Suppose there a exist a scalar function W(x) which, for real number $\in > 0$, satisfies the following properties for all x in the region $\|X\| < \epsilon$;

(a) W(x)>0; $x \neq 0$
(b) W (0) = 0
(c)W(x) has continuous partial derivatives with respect to all component of x
(d) $\frac{dW}{dt} \geq 0$
Then the system is unstable at the origin.

Direct Method of Liapunov & the Linear System:

In case of linear systems, the direct method of liapunov provides a simple approach to stability analysis. It must be emphasized that compared to the results presented, no new results are obtained by the use of direct method for the stability analysis of linear systems. However, the study of linear systems using the direct method is quite useful because it extends our thinking to nonlinear systems.
Consider a linear autonomous system described by the state equation

$$\dot{X} = AX \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (3.6)$$

The linear system is asymptotically stable in-the-large at the origin if and only if given any symmetric, positive definite matrix **Q,** there exists a symmetric positive definite matrix **P** which is the unique solution

$$A^T P + PA = -Q \ \ldots \ldots \ldots \ldots \ldots \ldots (3.7)$$

Proof
To prove the sufficiency of the result of above theorem, let us assume that a symmetric positive definite matrix **P** exists which is the unique solution of eqn.(3.8). Consider thescalar function.

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{P}\mathbf{x}$$

Note that $\quad V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$

$$V(\mathbf{0}) = \mathbf{0}$$

And

The time derivate of V(x) is

$$\dot{V}(X) = \dot{X}^T PX + X^T P\dot{X}$$

Using eqns. (3.6) and (3.7) we get

$$\begin{aligned} V(\mathbf{x}) &= \mathbf{x}^T\mathbf{A}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{A}\mathbf{x} \\ &= \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x} \\ &= -\mathbf{x}^T\mathbf{Q}\mathbf{x} \end{aligned}$$

Since **Q** is positive definite, V(x) is negative definite. Norm of x may be defined as

$$\|X\| = (X^T PX)^{1/2}$$

Then

$$V(X) = \|X\|^2$$
$$V(X) \to \infty \quad as \;\; \|X\| \to \infty$$

The system is therefore asymptotically stable in-the large at the origin.

In order to show that the result is also necessary, suppose that the system is asymptotically stable and P is negative definite, consider the scalar function

$$V(\mathbf{X}) = \mathbf{X^T P X} \;\; \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \; (3.8)$$

Therefore

$$\begin{aligned} \dot{V}(X) &= -[\dot{\mathbf{X}}^\mathbf{T}\mathbf{PX} + \mathbf{X}^\mathbf{T}\mathbf{P}\dot{\mathbf{X}}] \\ &= \mathbf{X^T Q X} \\ &> 0 \end{aligned}$$

There is contradiction since V(x) given by eqn. (3.8) satisfies instability theorem.

Thus the conditions for the positive definiteness of P are necessary and sufficient for asymptotic stability of the system of eqn. (3.6).

Methods of constructing Liapunov functions for Non linear Systems

As has been said earlier ,the liapunov theorems give only sufficient conditions on system stability and furthermore there is no unique way of constructing a liapunov function except in the case of linear systems where a liapunov function can always be constructed and both necessary and sufficient conditions Established .Because of this draw back a host of methods have become available in literature and many refinements have been suggested to enlarge the region in which the system is found to be stable. Since this treatise is meant as a first exposure of the student to the liapunov direct method, only two of the relatively simpler techniques of constructing a liapunov’s function would be advanced here.

Krasovskii’s method

Consider a system

$$\dot{x} = f(x);\; f(0) = 0$$

Define a liapunov function as

$$V = \mathbf{f^T P f} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (3.9)$$

Where P=a symmetric positive definite matrix.
Now

$$\dot{V} = \mathbf{\dot{f}^T P f} + \mathbf{f^T P \dot{f}} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots.. (3.10)$$

$$\dot{f} = \frac{\partial f}{\partial X} \frac{\partial X}{\partial t} = Jf$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots\dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots\dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots\dots. & \frac{\partial f_{2n}}{\partial x_n} \end{bmatrix}_{n\times n} \quad \text{is Jacobian matrix}$$

Substituting in eqn (3.10), we have

$$\begin{aligned} \dot{V} &= \mathbf{f^T J^T P f} + \mathbf{f^T P J f} \\ &= \mathbf{f^T}(\mathbf{J^T P} + \mathbf{PJ})\mathbf{f} \end{aligned}$$

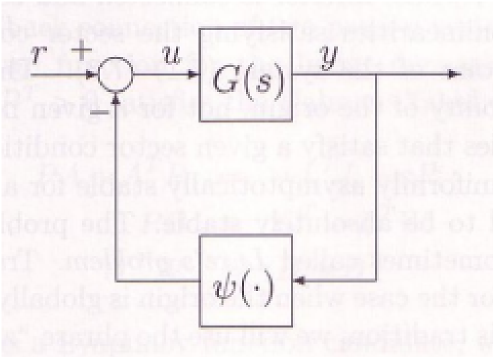
Let

$$\mathbf{Q} = \mathbf{J^T P} + \mathbf{PJ}$$

Since V is positive definite, for the system to be asymptotically stable, Q should be negative definite. If in addition $(X) \rightarrow \infty \;\; as \;\; \|X\| \rightarrow \infty$, the system is asymptotically stable in-the-large.

POPOV CRITERION

Many nonlinear physical systems can be represented as a feedback connection of a linear dynamical system and a nonlinear element.



The process for representing a system in this form depends on the particular system involved. For instance, in the case in which a control system's only nonlinearity is in the form of a relay or actuator/sensor nonlinearity, there is no difficulty in representing the system in this feedback form. In other cases, the representation may be less obvious. We assume that the extern input $r = 0$ and study the behavior of the unforced system. What is unique about this chapter is the use of the frequency response of the linear system, which builds on classical control tools like Nyquist plot and Nyquist criterion.
The system is said to be absolutely stable if it has a globally uniformly asymptotically stable equilibrium point at the origin for all nonlinearities in a given sector. The circle and Popov criteria give frequency-domain sufficient conditions for absolute stability in the form of strict positive realness of certain transfer functions. In the single-input-single-output case, both criteria can be applied graphically.

We assume the external input $r = 0$ and study the behavior of the unforced system represented by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$u = -\psi(y)$$

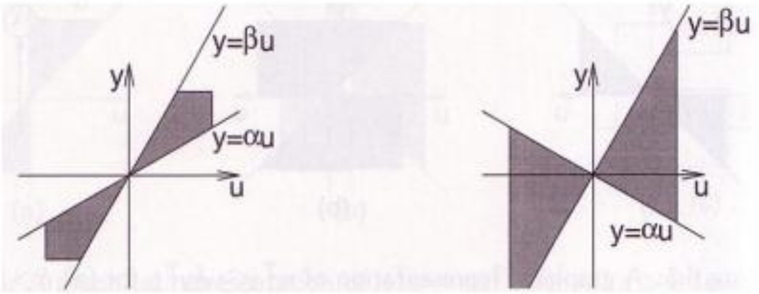
where $x \in \Re^n, u, y \in \Re^p, (A,B)$ is controllable, (A,C) is observable, and $\psi : [0,\infty) \times \Re^p \to \Re^p$ is a memoryless, possibly time-varying, nonlinearity, which is piecewise continuous in t and locally Lipschitz in y.

Definition 1.1: A memoryless function $h : [0,\infty) \to \Re$ is said to be to belong to the sector

1) $[0,\infty]$ if $u \cdot h(t,u) \geq 0$

2) $[\alpha,\infty]$ if $u\left[h(t,u)-\alpha u\right]\geq 0$

3) $[0,\beta]$ with $\beta>0$ if $h(t,u)\left[h(t,u)-\beta u\right]\leq 0$



Definition 1.2: The closed loop system is called absolutely stable in the sector $[0,\beta]$ if the origin is globally uniformly asymptotically stable for any nonlinearity in the given sector. It is absolutely stable in a finite domain if the origin is uniformly asymptotically stable.

1.1 Popov Criterion

We can apply the Popov criterion if the following conditions are satisfied:

1) The time-invariant nonlinearity $\psi:\Re\rightarrow\Re$ satisfies the sector condition $[0,\beta]$
2) The time-invariant nonlinearity $\psi:\Re\rightarrow\Re$ satisfies $\psi(0)=0$
3) $G(s)=\frac{1}{s^n}\frac{p(s)}{q(s)}$ with $\deg(p(s))<\deg(q(s))$
4) The poles of $G(s)$ are in the LHP or on the imaginary axis
5) The system is marginally stable in the singular case

Theorem 1.1: The closed loop system is absolutely stable if, $\psi\in[0,k],0<k<\infty$, and there exists a constant q such that the following equation is satisfied

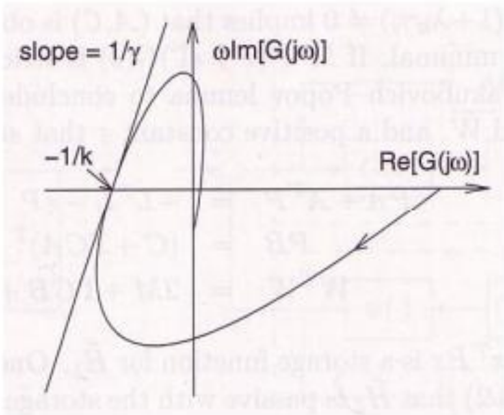
$$\text{Re}\left[G(j\omega)\right]-qj\omega\,\text{Im}\left[G(j\omega)\right]>-\frac{1}{k}\ ,\ \forall\omega\in[-\infty,\infty]$$

Graphical Interpretation

We define the Popov plot P

$$P(j\omega)=\left\{z=\text{Re}\left[G(j\omega)\right]+j\omega\,\text{Im}\left[G(j\omega)\right]\mid\omega>0\right\}$$

Then the closed loop system is absolutely stable if P lies to the right of the line that intercepts the point $-(1/k)+j0$ with a slope $1/q$.



Example 1.1: Consider the second-order system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - h(y)$$
$$y = x_1$$

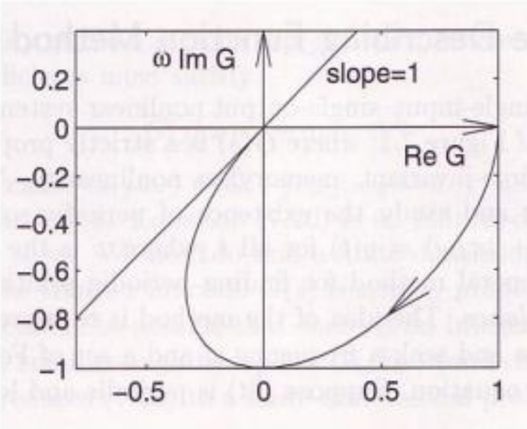
This system would fit the required form for using the Popov criterion if we took $\psi = h$, but the matrix A would not be Hurwitz. Adding and subtracting the term αy to the right-hand side of the second state equation, where $\alpha > 0$, and defining $\psi(y) = h(y) - \alpha y$, the system takes the required form, with

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \text{and } C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Assume that h belongs to a sector $[\alpha, \beta]$, where $\beta > \alpha$. Then ψ belongs to the sector $[0, k]$, where $k = \beta - \alpha$. The Popov condition takes the form

$$\frac{1}{k} + \frac{\alpha - \omega^2 + q\omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0 \ ,\ \forall \omega \in [-\infty, \infty]$$

For all finite positive values of α and k, this inequality is satisfied by choosing $q > 1$. Even at $k = \infty$, the foregoing inequality is satisfied for all $\omega \in (-\infty, \infty)$. Hence, the system is absolutely stable for all nonlinearities h in the sector $[\alpha, \infty]$, where α can be arbitrarily small.



The figure shows the Popov plot of $G(j\omega)$ for $\alpha = 1$. The plot is drawn only for $\omega \geq 0$, since $\text{Re}[G(j\omega)]$ and $\omega \text{Im}[G(j\omega)]$ are even functions of ω. The Popov plot asymptotically approaches the line through the origin of unity slope from the right side. Therefore, it lies to the right of any line of slope less than one that intersects the real axis at the origin and approaches it asymptotically as ω tends to ∞.

MODEL QUESTIONS

Module-3

Short Questions each carrying Two marks.

1. Explain how jump phenomena can occur in a power frequency circuit. Extend this concept to show that a ferro resonant circuit can be used to stabilize wide fluctuations in supply voltage of a.c. mains in a CVT(constant voltage transformer).
2. Explain various types of equilibrium points encountered in non-linear systems and draw approximately the phase plane trajectories.
3. Bring out the differences between Liapunov's stability criterion and Popov's stability criterion.
4. Explain what do you understand by limit cycle?

The figures in the right-hand margin indicate marks.

5. **(a)** Determine the describing function for the non-linear element described by $y=x^3$; where x=input and y= output of the non-linear element. **[5]**

(b) Draw the phase trajectory for the system described by the following differential equation

$$\frac{d^2X}{dt^2} + 0.6\,\frac{dX}{dt} + X = 0$$

With X(0)=1 and $\frac{dX}{dt}(0) = 0.$ **[5]**

6. Investigate the stability of the equilibrium state for the system governed by:

$$\frac{dX_1}{dt} = -3X_1 + X_2$$

$\frac{dX_2}{dt} = X_1 - X_2 - {X_2}^3$ **[7]**

7. Distinguish between the concepts of stability, asymptotic stability and global stability. **[3]**

8. Write short notes on **[3.5×6]**

(a) Signal stabilisation
(b) Delta method of drawing phase trajectories
(c) Phase plane portrait
(d) Jump resonance in non linear closed loop system
(e) Stable and unstable limit cycle
(f) Popov s stability criterion

9. **(a)** The origin is an equilibrium point for the pair of equations

$$\dot{X}_1 = aX_1 + bX_2$$
$$\dot{X}_2 = cX_1 + dX_2$$

Using Liapunov's theory find sufficient conditions on a, b, c and d such that the origin is asymptotically stable. **[8]**

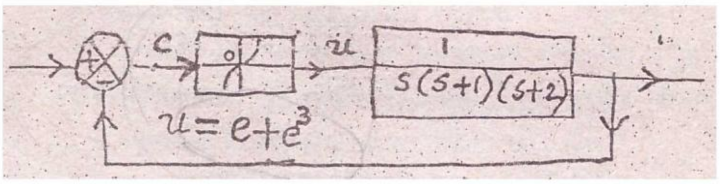
(b) A nonlinear system is described by

$$\frac{d^2x}{dt^2} + \sin x = 0.707$$

Draw the phase plane trajectory when the initial conditions are $x(0) = \frac{\pi}{3}, \dot{x}(0) = 0.$

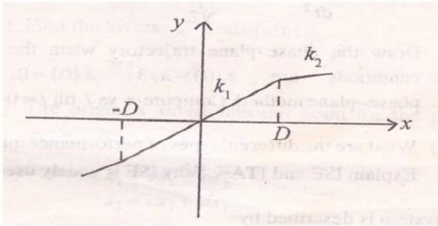
Use phase plane δ method. Compute x vs. t till t=0.1 sec. **[8]**

10. Determine the amplitude and frequency of oscillation of the limit cycle of the system shown in Figure below. Find the stability of the limit cycle oscillation. **[16]**



11. Write short notes on Popov's stability criterion and its geometrical interpretation. **[4]**

12. Derive the expression for describing function of the following non-linearity as shown in figure below. **[14]**



13. Describe Lyaponov's stability criterion. **[3]**

14. What do you mean by sign definiteness of a function? Check the positive definiteness of

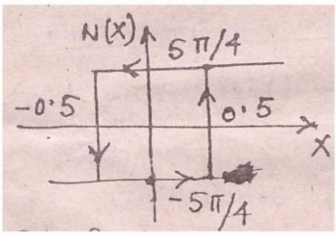
$$V(\hat{X}) = x_1^2 + \frac{2X_2^2}{1+X_2^2}$$ **[4]**

15. Distinguish between the concepts of stability, asymptotic stability & global stability. **[4]**

16. (a) What are singular points in a phase plane? Explain the following types of singularity with sketches: **[9]**

Stable node, unstable node, saddle point, stable focus, unstable focus, vortex.

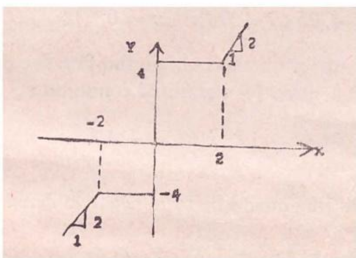
(b) Obtain the describing function of N(x) in figure below. Derive the formula used.



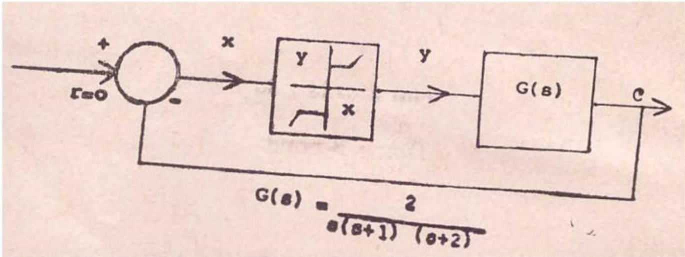
[6]

17. (a) Evaluate the describing function of the non linear element shown in figure below.

[6]



(b) This non linear element forms part of a closed loop system shown in fig below. Making use of the describing function analysis. Determine the frequency amplitude and stability of any possible self oscillation. **[10]**



18. (a) Explain of the method of drawing the trajectories in the phase plane using **[10]**

i) Lienard's construction

ii)Pell's method

(b) A second order non linear system is described by **[6]**

$\ddot{x}$ +25(1+0.1 $x^2)x = 0$

Using delta method obtain the first five points in the phase plane for initial condition

X(0)==1.8 $\dot{x}$(0)=-1.6

19. (a) In the following quadratic form negative definite? **[5]**

Q=-$x_1{}^2$-3$x_2{}^2$-11$x_3{}^2$ $+ 2x_1x_2$-4x_2x_3-2x_1x_3

(b) State and prove Liapunov's theorem for asymptotic stability of the system $\dot{x}$=A x

Hence show the following linear autonomous model **[6+5]**

$\dot{x}=\begin{bmatrix} 0 & 1 \\ -k & -a \end{bmatrix}$x

Is asymptotically stable if a>0,k>0.

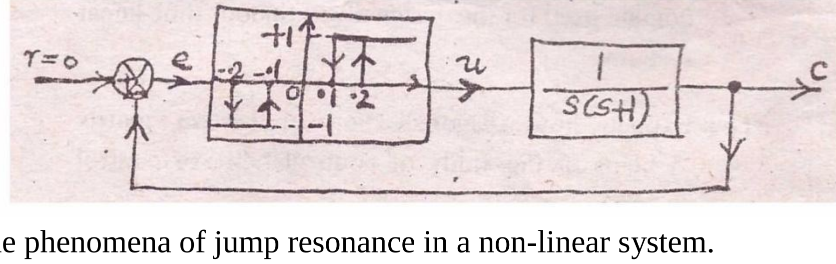
20. (a) Bring out the differnces between Liapunov's stability criterion and Popov's stability criterion. **[5]**

(b) $\dot{x_1}$ =x_2

$\dot{x_2}$=-0.9 sin x_1+0.5

Find the nature of singular points lying between $x_1 = 0^0$ to 180^0 **[10]**

21. A second order servo containing a relay with a dead zone and hysteresis is shown in fig below. Obtain the phase trajectory of the system for the initial condition e(0)=0.65 and $\dot{e}$(0)=0.does the system has a limit cycle? If so determine its amplitude and time period. **[15]**



22. Explain the phenomena of jump resonance in a non-linear system. **[4]**

23. Sometimes non-linear elements are intentionally introduced into control system. Give an example stating clearly the reason for the introduction of non-linear element. **[4]**

24. (a) A non-linear system is governed by

$$\frac{d^2x}{dt^2} + 8x - 4x^2 = 0$$

Determine the singular point s and their nature. Plot the trajectory passing through(($X_1 = 2, X_2 = 0$) without any approximation.

(b) What are the limitations of phase-plane analysis. **[12+3]**

25.(a) Find the describing function of the following type of non linearities. **[8]**

i)ideal on off relay

ii)ideal saturation

(b) Derive a Liapunov function for the defined by **[8]**

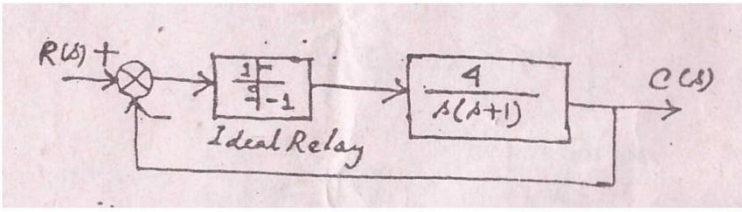
$$\dot{x_1} = x_2$$
$$\dot{x_2} = -3x_1^2 - 3\,x_2$$

Also check the stability of the system.

26.(a) Determine the singular points in the phase plane and sketch the plane trajectories for a system of characteristics equation

$$\frac{d^2x(t)}{dt^2} + 8x(t) - 4x^2(t) = 0$$ **[8]**

(b) A system described by the system shown in fig below



Will there be a limit cycle? If so determine its amplitude and frequency. **[8]**

MODULE-IV

OPTIMAL CONTROL SYSTEMS

Introduction:

There are two approaches to the design of control systems. In one approach we select the configuration of the overall system by introducing compensators to meet the given specifications on the performance. In other approach, for a given plant we find an overall system that meets the given specifications & then compute the necessary compensators.

The classical design based on the first approach, the designer is given a set of specifications in time domain or in frequency domain & system configuration. Compensators are selected that give as closely as possible, the desired system performance. In general, it may not be possible to satisfy all the desired specifications. Then, through a trial & error procedure, an acceptable system performance is achieved.

The trial & error uncertainties are eliminated in the parameter optimization method. In parameter optimization procedure, the performance specification consists of a single performance index. For a fixed system configuration, parameters that minimize the performance index are selected.

Parameter Optimization: Servomechanisms

The analytical approach of parameter optimization consists of the following steps:-

(i) Compute the performance index J as a function of the free parameters $K_1,K_2,\ldots,K_n$ of the system with fixed configuration:

$J=f(K_1,K_2,\ldots,K_n)$ ……………….(1)

(ii) Determine the solution set K_i of the equations

$$\frac{\partial J}{\partial K_i}=0; \quad i=1,2,\ldots\ldots..n \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Equation (2) give the necessary conditions for J to be minimum.

Sufficient conditions

From the solution set of equation(2), find the subset that satisfies the sufficient conditions which require that the Hessian matrix given below is positive definite.

$$H=\begin{bmatrix} \frac{\partial^2 J}{\partial K_1^2} & \frac{\partial^2 J}{\partial k_1 \partial k_2}\ldots\ldots & \frac{\partial^2 J}{\partial k_1 \partial k_n} \\ \frac{\partial^2 J}{\partial k_2 \partial k_1} & \frac{\partial^2 J}{\partial K_2^2}\ldots\ldots. & \frac{\partial^2 J}{\partial k_2 \partial k_n} \\ \ldots\ldots. & \ldots\ldots.. & \ldots\ldots\ldots \\ \frac{\partial^2 J}{\partial k_n \partial k_1} & \frac{\partial^2 J}{\partial k_n \partial k_2}\ldots\ldots & \frac{\partial^2 J}{\partial K_n^2} \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

Since $\frac{\partial^2 J}{\partial k_i \partial k_j} = \frac{\partial^2 J}{\partial k_j \partial k_i}$, the matrix H is always symmetric.

(iii) If there are two or more sets of K_i satisfying the necessary as well as sufficient conditions of minimization given by equations (2) & (3) respectively, then compute the corresponding J for each set.
The set that has the smallest J gives the optimal parameters.

Solution of Optimization Problem

The minimization problem will be more easily solved if we can express performance index interms of transform domain quantities.
The quadratic performance index, this can be done by using the Parseval's theorem which allows us to write

$$\int_0^{\infty} x^2(t)\,dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} X(s)X(-s)ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The values of right hand integral in equation(4) can easily be found from the published tables, provided that X(s) can be written in the form

$$X(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_0 s + \cdots + b_{n-1}s^{n-1}}{a_0 + a_0 s + \cdots + a_n s^n}$$

Where A(s) has zeros only in the left half of the complex plane.

$$J_1 = \frac{b_0^2}{2a_0 a_1}$$

$$J_2 = \frac{b_1^2 a_0 + b_0^2 a_2}{2a_0 a_1 a_2}$$

$$J_3 = \frac{b_2^2 a_0 a_1 + (b_1^2 - 2b_0 b_2)a_0 a_3 + b_0^2 a_2 a_3}{2a_0 a_3(-a_0 a_3 + a_1 a_2)}$$

Servomechanism or Tracking Problem

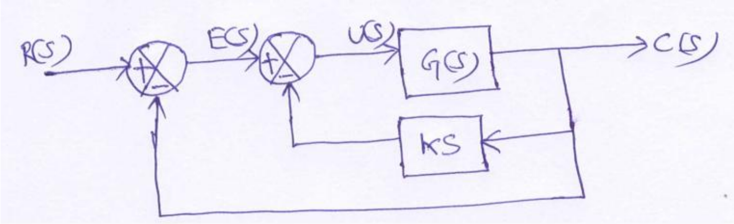
In servomechanism or tracking systems, the objective of design is to maintain the actual output c(t) of the system as close as possible to the desired output which is usually the reference input r(t) to the system.

We may define error e(t)=c(t) - r(t)

The design objective in a servomechanism or tracking problem is to keep error e(t) small. So performance index $J = \int_0^{\infty} e^2(t)\,dt$ is to be minimized if control u(t) is not constrained in magnitude.

EXAMPLE

Referring to the block diagram given below, consider $G(s) = \frac{100}{s^2}$ and $(s) = \frac{1}{s}$.
Determine the optimal value of parameter K such that $J = \int_0^\infty e^2(t)\, dt$ is minimum.



Solution

$$H(s) = \frac{G(s)}{1 + G(s)Ks} = \frac{100/s^2}{1 + \frac{100}{s^2}Ks} = \frac{100}{s + 100K}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + H(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1 + H(s)} = \frac{1/s}{1 + \frac{100}{s + 100K}} = \frac{s + 100k}{s^2 + 100ks + 100}$$

Here b_0=100K, b_1=1, a_0=100, a_1=100K, a_2=1

As E(s) is 2nd order

$$J_2 = \frac{{b_1}^2 a_0 + {b_0}^2 a_2}{2a_0 a_1 a_2} = \frac{1 + 100K^2}{200k}$$

$\frac{\partial J}{\partial K} = 0$ gives K=0.1 (necessary condition)

To check the sufficient condition, the Hessian matrix is $\frac{\partial^2 J}{\partial K^2} > 0$ (+ve definite)

∴ As necessary & sufficient condition satisfied, so the optimal value of the free parameter of the system is K=0.1.

Compensator design subject to constraints

The optimal design of servo systems obtained by minimizing the performance index

$$J = \int_0^\infty e^2(t)\, dt \ldots\ldots\ldots\ldots.(5)$$

may be unsatisfactory because it may lead to excessively large magnitudes of some control signals.

A more realistic solution to the problem is reached if the performance index is modified to account for physical constraints like saturation in physical devices. Therefore, a more realistic PI should be to minimize

$$J = \int_0^\infty e^2(t)\, dt \ldots\ldots\ldots\ldots.(6)$$

Subject to the constraint

$$max|u(t)| \leq M \ \ldots\ldots\ldots\ldots\ldots\ldots.(6a)$$

The constant M is determined by the linear range of the system plant.

If criterion given by equation(6) is used, the resulting optimal system is not necessarily a linear system; i.e in order to implement the optimal design, nonlinear &/or time-varying devices are required. So performance criterion given by equation (6) is replaced by the following quadratic PI:

$$J = \int_0^\infty [e^2(t) + \lambda u^2(t)]\, dt \ldots\ldots\ldots\ldots(7)$$

Where λ, a positive constant, is called the weighting factor.

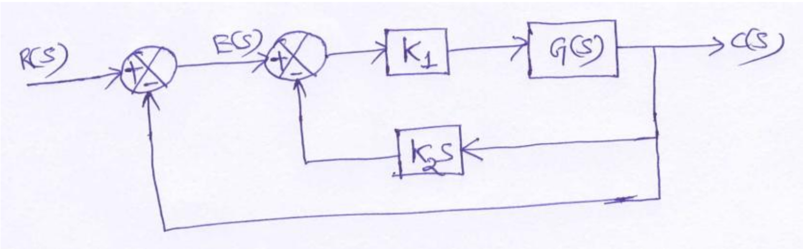
If λ is a small positive number, more weight is imposed on the error. As $\lambda \to 0$,the contribution of u(t) becomes less significant & PI reduces to $J = \int_0^\infty e^2(t)\, dt$. In this case, the magnitude of u(t) will be very large & the constraint given by equation (6a) may be violated. If $\to \infty$, performance criterion given by equation(7) reduces to

$$J = \int_0^\infty u^2(t)\, dt \ldots\ldots\ldots\ldots(8)$$

& the optimal system that minimizes this J is one with u=0.

From these two extreme cases, we conclude that if λ is properly chosen, then the constraint of (6a) will be satisfied.

EXAMPLE



$$G(s) = \frac{100}{s^2}\ ; \quad R(s) = \frac{1}{s}$$

Determine the optimal values of the parameters K_1 & K_2 such that

(i) $J_e = \int_0^\infty e^2(t)\, dt$ is minimized

(ii) $J_u = \int_0^\infty u^2(t)\, dt = 0.1$

Solution

$$\mathrm{H(s)} = \frac{\mathrm{K_1 G(s)}}{1 + \mathrm{K_1 G(s) K_2 s}} = \frac{\mathrm{K_1}\frac{100}{\mathrm{s^2}}}{1 + \frac{\mathrm{K_1}}{\mathrm{s^2}} 100\mathrm{K_2 s}} = \frac{\mathrm{K_1}\, 100/\mathrm{s}}{\mathrm{s} + \mathrm{K_1 K_2} 100}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + H(s)} = \frac{1}{1 + \frac{\mathrm{K_1} 100}{\mathrm{s^2} + \mathrm{K_1 K_2} 100}} = \frac{\mathrm{s^2} + \mathrm{K_1 K_2 s} 100}{\mathrm{s^2} + \mathrm{K_1 K_2 s} 100 + \mathrm{K_1} 100}$$

$$\Rightarrow E(s) = \frac{\frac{1}{s} s(s + \mathrm{K_1 K_2} 100)}{\mathrm{s^2} + \mathrm{K_1 K_2 s} 100 + \mathrm{K_1} 100} = \frac{s + \mathrm{K_1 K_2} 100}{\mathrm{s^2} + \mathrm{K_1 K_2 s} 100 + \mathrm{K_1} 100}$$

Here $b_0 = K_1 K_2 100$, $b_1 = 1$, $a_0 = K_1 100$, $a_1 = K_1 K_2 100$, $a_2 = 1$

As E(s) is of 2nd order, so PI

$$J_{e2} = \int_0^{\infty} e^2(t)\,dt = \frac{{b_1}^2 a_0 + {b_0}^2 a_2}{2a_0 a_1 a_2} = \frac{1 + 100\mathrm{K}_1 {K_2}^2}{200\mathrm{K}_1\mathrm{K}_2}$$

$$C(s) = \frac{100\mathrm{K}_1}{s(\mathrm{s}^2 + \mathrm{K}_1\mathrm{K}_2\mathrm{s}100 + \mathrm{K}_1 100)} = \frac{100}{s^2}U(s)$$

$$U(s) = \frac{s\mathrm{K}_1}{\mathrm{s}^2 + \mathrm{K}_1\mathrm{K}_2\mathrm{s}100 + \mathrm{K}_1 100}$$

Here $b_0=0$, $b_1=\mathrm{K}_1$, $a_0=\mathrm{K}_1 100$, $a_1=\mathrm{K}_1\mathrm{K}_2 100$, $a_2=1$

$$J_{u2} = \int_0^{\infty} u^2(t)\,dt = \frac{\mathrm{K}_1}{200\mathrm{K}_2}$$

The energy constraint on the system is thus expressed by the equation

$$J_u = \frac{\mathrm{K}_1}{200\mathrm{K}_2} = 0.1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(a)$$

The PI for the system is $J = J_e + \lambda J_u = \frac{1+100\mathrm{K}_1 {K_2}^2}{200\mathrm{K}_1\mathrm{K}_2} + \lambda \frac{\mathrm{K}_1}{200\mathrm{K}_2}$

$$\frac{\partial J}{\partial K_i} = 0 \;\; for\; i = 1,2\; gives$$

$$\lambda {K_1}^2 = 1 \ldots\ldots\ldots\ldots..(b)$$

$$100\mathrm{K}_1 {K_2}^2 - 1 - \lambda {K_1}^2 = 0 \ldots\ldots\ldots..(c)$$

Solving equation (a),(b),(c), we get $\lambda = 0.25,\ K_1 = 2,\ K_2 = 0.1$

The Hessian matrix $H = \begin{bmatrix} \frac{\partial^2 J}{\partial K_1^2} & \frac{\partial^2 J}{\partial k_1 \partial k_2} \\ \frac{\partial^2 J}{\partial k_2 \partial k_1} & \frac{\partial^2 J}{\partial K_2^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{100{\mathrm{K}_1}^3\mathrm{K}_2} & \frac{1-\lambda {K_1}^2}{200{K_1}^2{K_2}^2} \\ \frac{1-\lambda {K_1}^2}{200{K_1}^2{K_2}^2} & \frac{1}{\mathrm{K}_2} - \frac{100\mathrm{K}_1{K_2}^2-1}{100{\mathrm{K}_2}^3\mathrm{K}_1} \end{bmatrix}$

For $K_1 = 2,\ K_2 = 0.1$

$H = \begin{bmatrix} \frac{1}{80} & 0 \\ 0 & 5 \end{bmatrix}$ is positive definite

∴Therefore, $K_1 = 2,\ K_2 = 0.1$ satisfy the necessary as well as sufficient conditions for J to be minimum.

Output Regulator Problem

It is a special case of tracking problem in which r(t)=0. For zero input, the output is zero if all the initial conditions are zero. The response c(t) is due to non zero initial conditions that,in turn, are caused by disturbances. The primary objective of the design is to damp out the response due to initial conditions quickly without excessive overshoot & oscillations.

For example:- The disturbance torque of the sea causes the ship to roll. The response(roll angle Θ(t)) to this disturbance is highly oscillatory. The oscillations in the rolling motion are to be damped out quickly without excessive overshoot.

If there is no constraint on "control effort", the controller which minimizes the performance index.

$$J = \int_0^\infty (\theta(t) - \theta_d(t))^2 \, dt \dots\dots\dots\dots\dots\dots\dots (9)$$

Will be optimum.

$\theta_d(t) \rightarrow$ desired roll angle which is clearly zero.

Therefore, the problem of stabilization of ship against rolling motion is a regular problem. If for disturbance torque applied at $t=t_o$, the controller is required to regulate the roll motion which is finite time (t_f-t_o), a suitable performance criterion for design of optimum controller is to minimize

$$J = \int_{t_o}^{t_f} \theta^2(t) \, dt \dots\dots\dots\dots . (10)$$

Optimal Control Problems: State Variable Approach

Following steps are involved in the solution of an optimal control problem using state variable approach:

(i)Given plant in the of state equations

$\dot{X}(t) = AX(t) + Bu(t) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots . (11)$

A is constant matrix of size (n×n)

B is the constant matrix of size (n×m)

X(t) is the state vector of size (n×1)

U(t) is the control vector of size (m×1)

Find the control function u* which is optimal with respect to given performance criterion.

(ii) Realize the control function obtained from step (i).

The State Regulator Problem

When a system variable $x_1(t)$(the output) is required to be near zero, the performance measure is

$$J = \int_{t_o}^{t_f} {x_1}^2(t)\, dt$$

A performance index written interms of two state variables of a system would be then

$$J = \int_{t_o}^{t_f} ({x_1}^2(t) + {x_2}^2(t))dt$$

Therefore if the state x(t) of a system described by equation (11) is required to close to X_d=0, a design criterion would be to determine a control function that minimizes

$$J = \int_{t_o}^{t_f} (X^T X\,)\, dt$$

In practical, the control of all the states of the system is not equally important.

Example: In addition to roll angle $\theta(t)$ of aship,the pitch angle $\emptyset(t)$ is also required to be zero,so the PI gets modified to

$$J = \int_{t_o}^{t_f} (\theta^2(t) + \lambda\emptyset^2(t))dt$$

Where λ is a positive constant, a weighting factor.

The roll motion contributes much discomfort to passengers, in the design of passenger ship,the value of λ will be less than one.

A weighted PI is $J = \int_{t_o}^{t_f} (X^T QX\,)\, dt$

Where, Q=Error weighted matrix which is positive definite,real,symmetric,constant matrix.

The simplest form of Q is a diagonal matrix:

$$Q = \begin{bmatrix} q_1 & 0 \ldots\ldots & 0 \\ 0 & q_2 \ldots. & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \ldots\ldots & q_n \end{bmatrix}$$

The ith entry of Q represents the amount of weight the designer places on the constraint on state variable $x_i(t)$. The larger the value of q_i relative to other values of q,the more control effort is spent to regulate $x_i(t)$.

To minimize the deviation of the final state $X(t_f)$ of the system from the desired state $X_d = 0$, a possible performance measure is:

$$J = X^T(t_f)FX(t_f)$$

Where, F= Terminal cost weighted matrix which is positive definite, real, symmetric, constant matrix.

In the infinite time state regulator problem $(t_f \to \infty)$, the final state should approach the equilibrium state X=0; so the terminal constraint is no longer necessary..

The optimal design obtained by minimizing

$$J = X^T(t_f)FX(t_f) + \int_{t_o}^{t_f} X^T(t)QX(t)\,dt$$

The above PI is unsatisfactory in practice.

If PI is modified by adding a penalty term for physical constraints, then solution would be more realistic. So this is accomplished by introducing the quadratic control term in the PI

$$J = \int_{t_o}^{t_f} u^T(t)R\,u(t)\,dt$$

Where, R=Control weighted matrix which is positive definite,real,symmetric,constant matrix.

By giving sufficient weight to control terms, the amplitude of controls which minimize the overall PI may be kept within practical bound, although at the expense of increased error in X(t).

Continuous-time systems (state regulator problem)

Consider an LTI system

$$\dot{X}(t) = AX(t) + Bu(t) \ldots\ldots\ldots\ldots\ldots..(12)$$

Find the optimal control law u*(t), $t \in [t_0, t_f]$, where t_0 & t_f are specified initial & final times respectively, so that the optimal PI

$$J = \frac{1}{2}X^T(t_f)FX(t_f) + \frac{1}{2}\int_{t_o}^{t_f} [X^T(t)QX(t) + u^T(t)R\,u(t)]\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(13)$$

Is minimized, subject to initial state $x(t_0)=x_0$. t_f is fixed & given & $X(t_f)$ is free.

The matrices Q & F may be positive definite or semidefinite. We shall assume that, both Q & F are not simultaneously zero matrices to avoid trival solution.

Solution

(1) Solve matrix **differential Riccati equation**

$$\dot{P}(t) = -P(t)A - A^T P(t) - Q + P(t)BR^{-1}B^T P(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(14)$$

With final condition $P(t=t_f)=F$

Where P(t)=Riccati coefficient matrix,time varying matrix,symmetric positive definite matrix.

Riccati equation is nonlinear & for this reason, we usually cannot obtain closed form of solutions; therefore we must compute P(t) using digital computer.Numerical integration is carried out backward in time; from $t=t_f$ to $t=t_0$ with boundary condition $P(t_f)=F$.

(2) Obtain the optimal control u*(t) as

$$u^*(t) = -R^{-1}B^T P(t)X(t) = -K(t)X(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(15)$$

Where $K(t) = -R^{-1}B^T P(t)$ is called Kalman gain.

EXAMPLE

A first order system is described by the differential equation $\dot{x}(t) = 2x(t) + u(t)$.
It is desired to find the control Law that minimizes the PI

$$J = \frac{1}{2}\int_{t_o}^{t_f}\left(3x^2 + \frac{1}{4}u^2\right)dt \;,\quad t_f = 1\,sec$$

Solution
Comparing the state equation with equation(12), A=2, B=1
Comparing PI with equation(13), we get $F = 0, R = \frac{1}{4}, Q = 3$
As there is one state variable so P(t)=p(t), matrix reduces to scalar function.
The matrix differential Riccatii equation becomes scalar differential equation

$$\dot{P}(t) = -P(t)A - A^T P(t) - Q + P(t)BR^{-1}B^T P(t)$$
$$= -P(t)\times 2 - 2P(t) - 3 + P(t)\times 1\times 4\times 1\times P(t)$$
$$= -4P(t) - 3 + 4P^2(t)$$

With boundary condition $P(t_f) = F = 0$
Solution is obtained by numerical integration backward intime

$$\int_{t_f}^{t}\dot{P}(t)\,dt = \int_{t_f}^{t}\left(-4P(t) - 3 + 4P^2(t)\right)dt$$

$$\Rightarrow \int_{t_f}^{t} dP(t) = \int_{t_f}^{t} \left[4\left(P(t)-\frac{3}{2}\right)\left(P(t)+\frac{1}{2}\right)\right] dt$$

Separating variables

$$\Rightarrow \int_{t_f}^{t} \frac{1}{4\left(P(t)-\frac{3}{2}\right)\left(P(t)+\frac{1}{2}\right)} dP(t) = \int_{t_f}^{t} dt$$

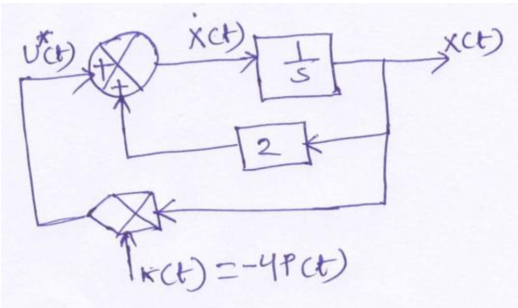
$$\Rightarrow \frac{1}{8}\left[ln\left\{\frac{\left(P(t)-\frac{3}{2}\right)}{-\frac{3}{2}} \Big/ \frac{\left(P(t)+\frac{1}{2}\right)}{\frac{1}{2}}\right\}\right] = t - t_f$$

$$\Rightarrow P(t) = \frac{\frac{3}{2}\left(1-e^{8(t-t_f)}\right)}{1+3e^{8(t-t_f)}}$$

The Optimal control law

$$u^*(t) = -R^{-1}B^T P(t)X(t) = -4P(t)x(t)$$

The block diagram for Optimal control system is



Infinite time regulator problem

In equation (13), if the terminal time t_f ,is not constrained, the PI is

$$J = \frac{1}{2}\int_{t_o}^{t_f} [X^T(t)QX(t) + u^T(t)R\,u(t)]\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(16)$$

Salient points of infinite time regulator problem:

(1) When $t_f \rightarrow \infty$, $X(\infty) \rightarrow 0$ for the optimal system to be stable.Therefore the terminal penalty term has no significance; consequently it does not appear in J i.e we set F=0 in general quadratic PI.

(2) As $t_f \rightarrow \infty$ & $F = 0 = P(t_f)$

$\Rightarrow \lim_{t_f \rightarrow \infty} P(t_f) = \bar{P}$ which is a constant matrix

(3)As $\bar{P}$ is a constant ,so $\dot{P}(t) = 0$, substituting this in equation(14),we get

$$-\overline{P}A - A^T\overline{P} - Q + \overline{P}BR^{-1}B^T\overline{P} = 0$$

The above equation is known as **algebraic matrix Riccatic equation(ARE) or reduced matrix Riccati equation.**

(4) Solve ARE to get $\overline{P}$,then the optimal control law is given by

$$u^*(t) = -R^{-1}B^T\overline{P}X(t) = -KX(t)$$

Where $K(t) = -R^{-1}B^T\overline{P}$ is called Kalman gain.

The optimal law is implemented using time invariant Kalman gain in contrast to the finite-time case.

(5) The optimal value of PI is

$$J^* = \frac{1}{2}X^T(0)\,\overline{P}\,X(0)$$

<u>EXAMPLE</u>

Obtain the control Law which minimizes the performance index

$$J = \int_0^\infty ({x_1}^2 + u^2)\,dt$$

For the system

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

<u>Solution</u>

$$A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, B = \begin{bmatrix}0\\ 1\end{bmatrix}, Q = \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}, R = 2$$

ARE

$$-\overline{P}A - A^T\overline{P} + \overline{P}BR^{-1}B^T\overline{P} - Q = 0$$

$$-\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix} + \begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix} - \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Simplifying

$$-\frac{{p_{12}}^2}{2} + 2 = 0$$

$$p_{11} - \frac{p_{12}p_{22}}{2} = 0$$

$$-\frac{{p_{22}}^2}{2} + 2p_{12} = 0$$

For $\overline{P}$ to be positive definite matrix we get the solution $\overline{P} = \begin{bmatrix}2\sqrt{2} & 2\\ 2 & 2\sqrt{2}\end{bmatrix}$,

The Optimal Control Law is given by

$$u^*(t) = -R^{-1}B^T\bar{P}\,X(t) = -\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}2\sqrt{2} & 2\\ 2 & 2\sqrt{2}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = -x_1(t) - \sqrt{2}x_2(t)$$

It can be easily verified that closed loop system is asymptotically stable.(Though Q is positive definite)

The Output Regulator Problem

In the state regulator problem, we are concerned with making all the components of the state vector X(t) small.In the output regulator problem on the other hand, we ere concerned with making the components of the output vector small.

Consider an observable controlled process described by the equations

$$\dot{X}(t) = AX(t) + Bu(t) \dots \dots \dots \dots \dots . . (17)$$

$$Y(t) = CX(t)$$

Find the optimal control law u*(t),$t \in [t_0, t_f]$, where t_0 & t_f are specified initial & final times respectively, so that the optimal PI

$$J = \frac{1}{2}Y^T(t_f)F\,Y(t_f) + \frac{1}{2}\int_{t_o}^{t_f}[Y^T(t)QY(t) + u^T(t)R\,u(t)]\,dt \dots \dots \dots \dots \dots \dots \dots (18)$$

Is minimized,subject to initial state $x(t_0)=x_0$.

Tracking Problem

Consider an observable controlled process described by the equation(17). Suppose that the vector Z(t) is the desired output.

The error vector e(t)=Z(t)-Y(t)

Find the optimal control law u*(t),$t \in [t_0, t_f]$, where t_0 & t_f are specified initial & final times respectively, so that the optimal PI

$$J = \frac{1}{2}e^T(t_f)F\,e(t_f) + \frac{1}{2}\int_{t_o}^{t_f}[e^T(t)Qe(t) + u^T(t)R\,u(t)]\,dt$$

Is minimized .

Output Regulator as state regulator Problem

If the controlled process given by equation(17) is observable then, we can reduce the output regulator problem to the state regulator problem.

Substituting Y(t)=CX(t) in the PI given by equation (18), we get

$$J=\frac{1}{2}X^T(t_f)C^TFCX(t_f)+\frac{1}{2}\int_{t_o}^{t_f}[X^T(t)C^TQCX(t)+u^T(t)R\,u(t)]\,dt$$

For this PI , a unique optimal control exists & is given by

$$u^*(t)=-R^{-1}B^TP(t)X(t)=-K(t)X(t)$$

Where P(t) is the solution of the matrix Riccati equation given by:

$$\dot{P}(t)=-P(t)A-A^TP(t)-C^TQC+P(t)BR^{-1}B^TP(t)\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(19)$$

With boundary condition $P(t_f)=C^TFC$

The Tracking Problem

Here we shall study a class of tracking problems which are reducible to the form of the output regulator problem.

Consider an observable controlled process described by the equations

$$\dot{X}(t)=AX(t)+Bu(t)$$

$$Y(t)=CX(t)$$

It is desired to bring & keep output Y(t) close to the desired output r(t).

We define error vector e(t)=Y(t)-r(t)

Find the optimal control law u*(t),$t\in[t_0,t_f]$, where t_0 & t_f are specified initial & final times respectively, so that the optimal PI

$$J=\frac{1}{2}e^T(t_f)F\,e(t_f)+\frac{1}{2}\int_{t_o}^{t_f}[e^T(t)Qe(t)+u^T(t)R\,u(t)]\,dt$$

Is minimized .

To reduce this problem to the form of output regulator problem,we consider only those r(t) that can be generated by arbitrary initial conditions Z(0) in the system.

$$\dot{Z}(t)=AZ(t)$$

$$r(t)=CZ(t)$$

The matrices A & C are same as those of the plant. Now define a new variable W=X-Z

Then

$$\dot{W}(t) = AW(t) + Bu(t)$$

$$e(t) = CW(t)$$

Applying results of the output regulator problem gives immediately that the optimal control for the tracking problem under consideration is

$$u^*(t) = -R^{-1}B^TP(t)W(t) = -K(t)[X - Z]$$

Where P(t) is the solution of the Riccati equation(19).

Parameter Optimization: Regulators

Solution of control problem when some elements of feedback matrix K are constrained

Consider completely controllable process

$$\dot{X}(t) = AX(t) + Bu(t)$$

The PI is $J = \frac{1}{2}\int_0^\infty [X^T(t)QX(t) + u^T(t)R\,u(t)]\,dt$

Optimal control is linear combination of the state variables u=KX(t)

With the above feedback law,closed loop system is described by

$$\dot{X}(t) = AX(t) + BKX(t) = (A + BK)X(t)$$

Solution

(1) Determine elements of P as functions of the elements of the feedback matrix K from the equations given below

$$(A + BK)^TP + P(A + BK) + K^TRK + Q = 0 \ldots\ldots\ldots\ldots\ldots\ldots.(20)$$

(2) Find PI which is given as

$$J = \frac{1}{2}X^T(0)P\,X(0) \ldots\ldots\ldots\ldots\ldots\ldots.(21)$$

If $K_1, K_2, \ldots, K_n$ are the free elements of matrix P, we have

$$J = f(K_1, K_2, \ldots, K_n) \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

(3) The necessary & sufficient conditions for J to be minimum are given by

$$\frac{\partial J}{\partial K_i} = 0; \quad i = 1,2,\ldots\ldots..n \qquad (necessary\ condition)$$

Hessian matrix is positive definite (sufficient condition)

Solution set K_i of equation(22) satisfies necessary & sufficient condition is obtained which gives the suboptimal solution to the control problem. Of course, K_i must satisfy the further constraint that the closed-loop system be asymptotically stable. If all the parameters of P are free, the procedure above will yield an optimal solution.

Special Case

Where the PI is independent of the control u, we have

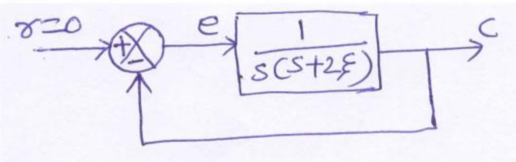
$$J = \frac{1}{2}\int_0^{\infty} [X^T(t)QX(t)]\, dt$$

In this case the matrix P is obtained from the equation(20) by substituting R=0 resulting in the modified matrix equation

$$(\mathrm{A}+\mathrm{BK})^{\mathrm{T}}\mathrm{P}+\mathrm{P}(\mathrm{A}+\mathrm{BK})+\mathrm{Q} \;=0 \ldots\ldots\ldots\ldots\ldots\ldots.(23)$$

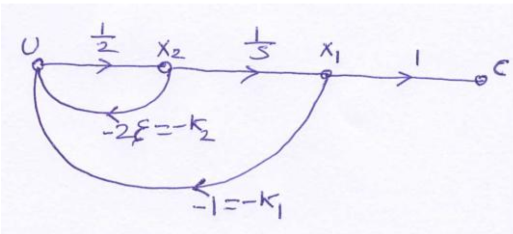
EXAMPLE

Consider the second order system, where it is desired to find optimum ζ which minimizes the integral square error i.e J=$\int_0^{\infty} e^2(t)\, dt$ for the initial conditions c(0)=1,$\dot{c}(0)=0$



Solution

The problem is reframed in the state form with one of obtaining feedback control law with constraint K_1=1



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u, \qquad x_1(0)=1, x_2(0)=0$$

$$\mathrm{u} = -[\mathrm{K}_1 \quad \mathrm{K}_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Now $J = \int_0^{\infty} e^2(t)\, dt = \int_0^{\infty} {x_1}^2\, dt \quad since\ e=-c=-x_1$

Therefore $Q = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

Substituting the values in equation(23)

$$\begin{bmatrix}0 & -1\\0 & -K_2\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\p_{12} & p_{22}\end{bmatrix}+\begin{bmatrix}p_{11} & p_{12}\\p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}0 & 1\\-1 & -K_2\end{bmatrix}+\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

Solving we get $P=\begin{bmatrix}\frac{1+K_2^2}{K_2} & 1\\1 & \frac{1}{K_2}\end{bmatrix}$

PI $J=\frac{1}{2}X^T(0)P\,X(0)=\frac{1+K_2^2}{2K_2}$

J to be minimum, $\frac{\partial J}{\partial K_2}=\left(\frac{1}{2}-\frac{1}{2K_2^2}\right)=0$

$$\Rightarrow K_2=1$$

$\frac{\partial^2 J}{\partial K_2^2}=\frac{1}{K_2^3}>0$, this is satisfied for $K_2=1$

Therefore, optimal value of parameter $K_2=1$. Since $K_2=2\xi$, $\Rightarrow \xi=0.5$ minimizes integral square error for the given initial conditions.

It can be easily verified that the suboptimal control derived above results in a closedloop system which is asymptotically stable.

The control given by $J=\frac{1}{2}X^T(0)P\,X(0)$ will vary from one x(0) to another. For practical reasons, it is desirable to have one control irrespective of what x(0) is.

One way to solve this problem is to assume that x(0) is a random variable, uniformly distributed on the surface of the n-dimensional unit sphere.

$$E[X(0)X^T(0)]=\frac{1}{n}I$$

New PI $\hat{J}=E[J]=E\left[\frac{1}{2}X^T(0)P\,X(0)\right]=\frac{1}{2n}tr[p]$

$$tr[p]=trace\ of\ P=sum\ of\ all\ diagonal\ elements\ of\ P$$

The parameter K_2 that optimizes $\hat{J}$ is $\sqrt{2}$.

Therefore

$$u=-\begin{bmatrix}1 & \sqrt{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

is suboptimal control law that is independent of initial conditions.

INTRODUCTION TO ADAPTIVE CONTROL

To implement high performance control systems when the plant dynamic characteristics are poorly known or when large & unpredictable variations occur, a new class of control systems called nonlinear control systems have evolved which provide potential solutions. Four classes of nonlinear controllers for this purposes are

(1)Robust controllers (2) Adaptive controllers (3) Fuzzy logic controllers (4) Neural controllers

Adaptive control

An adaptive controller is a controller that can modify its behaviour in response to changes in dynamics of the process & the disturbances. One of the goals of adaptive control is to compensate for parameter variations, which may occur due to nonlinear actuators, changes in the operating conditions of the process, & non-stationary disturbances acting on the process.

An adaptive controller is a controller with adjustable parameters & a mechanism for adjusting the parameters.

An adaptive control system may be thought of as having two loops. One loop is a normal feedback with the process(plant) & controller. The other loop is a parameter adjustment loop. The block diagram of an adaptive system is shown below. The parameter adjustment loop is often slower than the normal feedback loop.

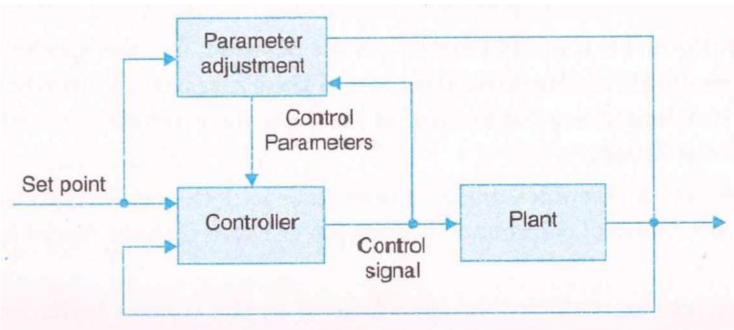


Fig: Adaptive Controller

There are two main approaches for designing adaptive controllers. They are

(1) Model Reference Adaptive control Method

(2) Self-Tuning method

(1) Model Reference Adaptive control (MRAC)

The MRAC system is an adaptive system in which the desired performance is expressed in terms of a reference model, which gives the desired response signal.

Marc is composed of four parts.

(a) a plant containing unknown parameters

(b) A reference model for compactly specifying the desired output of the control system.

(c) a feedback control law containing adjustable parameters.

(d) parameter adjustment loop is called as outer loop..

The ordinary feedback loop is known as the inner loop.

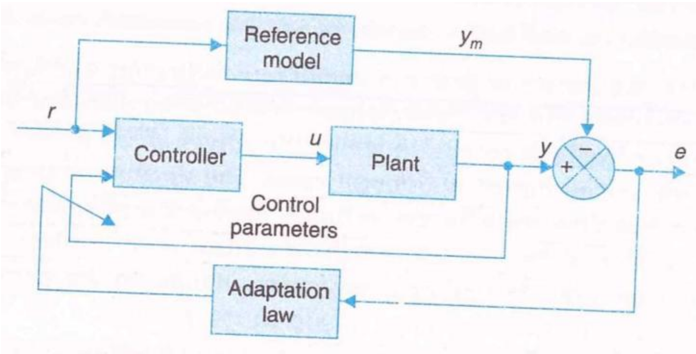


Fig: Model Reference Adaptive controller

(2) Self-Tuning control

A general architecture for the self tuning control is shown below.

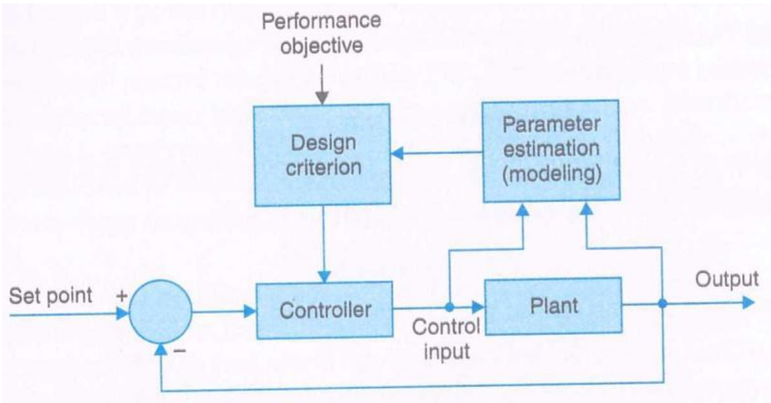


Fig: A general configuration of Self -Tuning controller

Self tuning control has two essential components. They are

(1) Parameter estimation   (2) control law

(1) Parameter estimation: Parameter estimation is performed online. The model parameters are estimated based on the measureable process input, process output & the state signals. A number of recursive parameter estimation schemes are employed for self tuning control. The most popular scheme is the recursive least square estimation method.

(2) Control law: The control law is derived based on control performance criterion optimization. Since the parameters are estimated on-line, the calculation of control law is based on a procedure called certainty equivalent in which the current parameter estimates are accepted while ignoring their uncertainties. This approach of designing controller using estimated parameters of the transfer function of the process is known as indirect self-tuning method.

MODEL QUESTIONS

Module-4

The figures in the right-hand margin indicate marks.

1. What do you understand about parameter optimisation of Regulators? **[2]**

2. Find the control law which minimizes the performance index

$$J = \int_0^\infty ({X_1}^2 + U^2)\, dt$$

For the system

$$\begin{bmatrix} \dot{X_1} \\ \dot{X_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$ **[10]**

3. Write short notes on. **[3.5×2]**

(a) State regulator problem

(b) Pontryagin's minimum principle

4. A system is described by

$$\dot{X_1} = X_2$$
$$\dot{X_2} = -2X_1 - 3X_2 + u$$

Determine the optimal control law $u_{opt}(t)$ such that the following performance index is minimised

$$J = \frac{1}{2}\int_0^\infty ({x_1}^2 + {x_2}^2 + u^2)\, dt$$

Derive the formula used. **[8+8]**

5. What are the different types of performance indices? Explain ISE & ITAE. Why ISE is widely used? **[8]**

6.(a) Explain the following error performance indices:

ISE, ISTE, IAE, ITAE **[6]**

(b) Determine the optimal controller to minimize

$$J = \int_0^\infty (y^2 + u^2)\, dt$$

for the process described by $\frac{dy}{dt} + y = u$ **[9]**

7. Consider a system described by

$\dot{X} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u; x_1(0) = x_2(0) = 1$

Where u $= -x_1 - kx_2$

(i) Find the value of k so that

$J = \frac{1}{2}\int_0^\infty ({x_1}^2 + {x_2}^2) dt$ *is minimised,*

(ii) Find the minimum value of J

(iii) Find sensitivity of J with respect to k **[15]**

8. A linear autonomous system is described in the state equation

$$\dot{X} = \begin{bmatrix} -4\dot{K} & 4K \\ 2K & -6K \end{bmatrix} X$$

Find restriction on the parameter k to guarantee stability of the system. **[15]**

9. A first order system is described by the differential equation

$$X(t) = 2\dot{X}(t) + u(t)$$

Find the control law that minimises the performance index

$$J=\frac{1}{2}\int_0^{t_f}(3X^2 + \frac{1}{4}u^2)\, dt$$

When t_f =1 second **[15]**

10.(a) What do you understand about parameter optimisation of regulator? **[6]**

(b) Find the control laws which minimises the performance index **[10]**

$$J=\int_0^{\infty}({x_1}^2 + u^2)\, dt$$

For the system

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$