MCE2121

ADVANCED FLUID MECHANICS

LECTURE NOTES

Module-II

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Course Content

Module II

Viscous Flow and Boundary Layer Theory: Study of Local Behavior, Differential Approaches in Analysis of Viscous Flows, Equations of Motion of a Viscous Flow, Navier – Stokes Equations, Exact and Approximate Solution of N – S Equations, Hele – Shaw Flow, Creeping Flow past a Sphere, Boundary Layer Concept, Prandtl's Boundary Layer Equations, Laminar Boundary Layer Along a Flat Plate, Integral Equation, Blassius Solution.

Lecture Note 1

Boundary Layer Theory

Introduction

- The **boundary layer** of a flowing fluid is **the thin layer close to the wall**
- In a flow field, **viscous stresses are very prominent within this layer**.
- Although the layer is thin, it is very important to know the details of flow within it.
- The **main-flow velocity** within this layer **tends to zero** while approaching the wall **(no-slip condition)**.
- Also the gradient of this velocity component in a direction normal to the surface is large as compared to the gradient in the streamwise direction.

Boundary Layer Equations

- In 1904, **Ludwig Prandtl,** the well known German scientist, introduced the concept of boundary layer and **derived the equations for boundary layer flow** by correct reduction of Navier-Stokes equations.

- He hypothesized that **for fluids having relatively small viscosity**, **the effect of internal friction in the fluid is significant only in a narrow region surrounding solid boundaries or bodies over which the fluid flows.**

- Thus, close to the body is the boundary layer where **shear stresses exert an increasingly larger effect** on the fluid **as one moves from free stream towards the solid boundary.**

- However, **outside the boundary layer where the effect of the shear stresses on the flow is small compared to values inside the boundary layer (since the velocity gradient** $\partial u / \partial y$ **is negligible)**,---------

1. the fluid particles experience **no vorticity** and therefore**,**
2. the flow is similar to a **potential flow.**

- Hence, the ***surface*** **at the boundary layer interface** is a rather fictitious one, that **divides rotational and irrotational flow.** Fig 1 shows Prandtl's model regarding boundary layer flow.

- Hence with the exception of the immediate vicinity of the surface, the flow is frictionless (inviscid) and the velocity is *U* (the potential velocity).

- In the region, very near to the surface (in the thin layer), there is friction in the flow which signifies that the fluid is retarded until it adheres to the surface (**no-slip condition).**

- The transition of the mainstream velocity from zero at the surface (with respect to the surface) to full magnitude takes place across the boundary layer.

About the boundary layer

- Boundary layer **thickness is** δ which is a **function of** the coordinate direction ***x*** .
- The thickness is considered to be **very small compared to the characteristic length *L*** of the domain.
- In the normal direction, **within this thin layer**, the gradient $\partial u / \partial y$ **is very large compared to** the gradient in the flow direction $\partial u / \partial x$ **.**

Now we take up the Navier-Stokes equations for : steady, two dimensional, laminar, incompressible flows.

Considering the Navier-Stokes equations together with the equation of continuity, the following dimensional form is obtained.

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \qquad (1)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \qquad (2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (3)$$

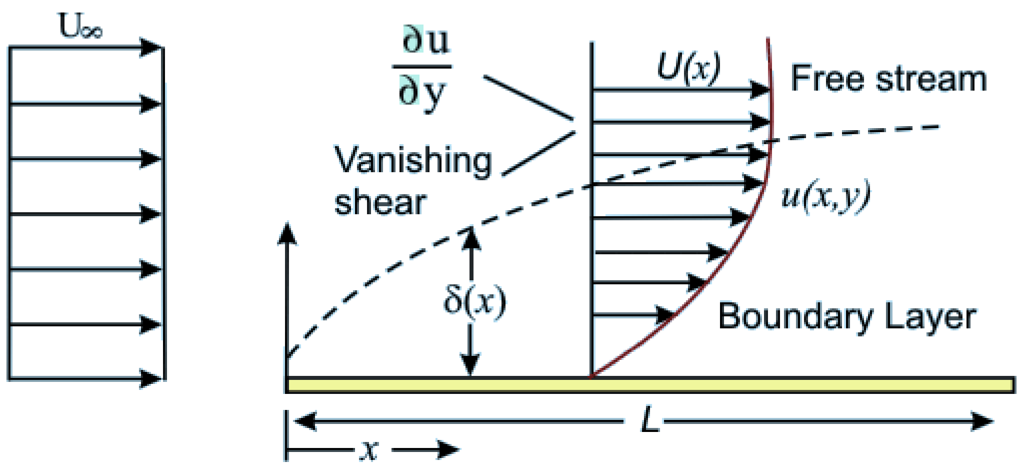

Fig 1 Boundary layer and Free Stream for Flow Over a flat plate

- u - velocity component along x direction.
- v - velocity component along y direction
- p - static pressure
- ρ - density.

- μ - dynamic viscosity of the fluid

- The equations are now non-dimensionalised.
- The **length and the velocity scales** are chosen as ***L*** **and** U_∞ **respectively.**
- The non-dimensional variables are:

$$u^* = \frac{u}{U_\infty}, v^* = \frac{v}{U_\infty}, p^* = \frac{p}{\rho U_\infty^2}$$

$$x^* = \frac{x}{L}, y^* = \frac{y}{L}$$

where U_∞ is the dimensional free stream velocity and the pressure is non-dimensionalised by twice the dynamic pressure $p_d = (1/2)\rho U_\infty^2$.

Using these non-dimensional variables, the Eqs (1) to (3) become

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}}\left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right] \quad (4)$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}}\left[\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right] \quad (5)$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \quad (6)$$

where the Reynolds number,

$$\mathrm{Re} = \frac{\rho U_\infty L}{\mu}$$

Order of Magnitude Analysis

- Let us examine what happens to the *u* velocity as we go across the boundary layer. At the **wall** the *u* velocity is **zero** [with respect to the wall and absolute zero for a stationary wall (which is normally implied if not stated otherwise)]. The value of *u* on the **inviscid side**, that is on the free stream side beyond the boundary layer is ***U.*** For the case of external flow over a flat plate, this ***U*** is equal to U_∞.
- Based on the above, we can identify the following scales for the boundary layer variables:

Variable	*Dimensional scale*	*Non-dimensional scale*
u	U_∞	1
x	L	1
y	δ	$\varepsilon = \delta / L$

- The **symbol ε describes a value much smaller than 1.**
- Now we analyse equations 4 - .6, and look at the order of magnitude of each individual term

Eq 6 - the continuity equation

One **general rule** of incompressible fluid mechanics is that **we are not allowed to drop any term from the continuity equation**.

- From the scales of boundary layer variables, the derivative $\partial u^* / \partial x^*$ is of the order 1.
- The second term in the continuity equation $\partial v^* / \partial y^*$ should also be of the order 1.The reason being v^* has to be of the order ϵbecause y^* becomes $\varepsilon (= \delta / L)$ at its maximum.

Eq 4 - x direction momentum equation

- Inertia terms are of the order 1.
- $\partial^2 u / \partial x^2$ is of the order 1
- $\partial^2 u / \partial y^2$ is of the order $(1/\varepsilon^2)$.

However after multiplication with 1/Re, the sum of the two second order derivatives should produce at least one term which is of the same order of magnitude as the inertia terms. This is possible only if the Reynolds number (Re) is of the order of $(1/\varepsilon^2)$.

- It follows from that $-\partial p^* / \partial x^*$ will not exceed the order of 1 so as to be in balance with the remaining term.

- Finally, Eqs (4), (5) and (6) can be rewritten as

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}} \left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} \right] \qquad (7)$$

$$(1)\frac{(1)}{(1)} \qquad (\varepsilon)\frac{(1)}{(\varepsilon)} = (1) \qquad (\varepsilon^2)\left[\frac{(1)}{(1)} + \frac{1}{(\varepsilon^2)}\right]$$

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}}\left[\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right] \qquad (8)$$

$$(1)\frac{(\varepsilon)}{(1)} \qquad (\varepsilon)\frac{(\varepsilon)}{(\varepsilon)} = (?) \qquad (\varepsilon^2)\left[\frac{(\varepsilon)}{(1)} + \frac{\varepsilon}{(\varepsilon^2)}\right]$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{(1)}{(1)} \qquad \frac{(\varepsilon)}{(\varepsilon)}$$

As a consequence of the order of magnitude analysis, $\partial^2 u^* / \partial x^{*2}$ can be dropped from the x direction momentum equation, because on multiplication with $1/Re$ it assumes the smallest order of magnitude.

Eq 5 - *y* direction momentum equation.

- All the terms of this equation are of a smaller magnitude than those of Eq. (.4).
- This equation can only be balanced if $\partial p^* / \partial y^*$ is of the same order of magnitude as other terms.
- Thus they momentum equation reduces to

$$\frac{\partial p^*}{\partial y^*} = O(\varepsilon) \qquad (8)$$

- This means that the **pressure across the boundary layer does not change**. The **pressure is impressed on the boundary layer**, and its value is determined by hydrodynamic considerations.
- This also implies that the **pressure *p* is only a function of *x***. The pressure forces on a body are solely **determined by the inviscid flow outside the boundary layer.**
- The application of Eq. (28.4) at the outer edge of boundary layer gives

$$u^*\frac{\partial u^*}{\partial x^*} = -\frac{\partial p^*}{\partial x^*} \qquad (9)$$

In dimensional form, this can be written as

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (10)$$

On integrating Eq (28.8b) the well known Bernoulli's equation is obtained

$$p + \frac{1}{2}\rho U^2 = \text{a constant} \qquad (11)$$

- Finally, it can be said that by the order of magnitude analysis, the Navier-Stokes equations are simplified into equations given below.

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u^*}{\partial y^{*2}} \qquad (12)$$

-

$$\frac{\partial p^*}{\partial y^*} = 0 \qquad (13)$$

-

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \qquad (14)$$

-
- These are known as Prandtl's boundary-layer equations.

The available boundary conditions are:

Solid surface $at\ y^* = 0,\ u^* = 0 = v^*$

or $at\ y = 0,\ u = 0 = v$ (15)

Outer edge of boundary-layer

$$at\ y^* = (\varepsilon) = \frac{\delta}{L},\ u^* = 1$$

or $at\ y = \delta, u = U(x)$ (16)

- The unknown pressure *p* in the x-momentum equation can be determined from Bernoulli's Eq. (28.9), if the inviscid velocity distribution *U(x)* is also known.

We solve the Prandtl boundary layer equations for $u^*(x,y)$ and $v^*(x,y)$ with *U* obtained from the outer inviscid flow analysis. The equations are solved by commencing at the leading edge of the body and moving downstream to the desired location

- it allows the no-slip boundary condition to be satisfied which constitutes a significant improvement over the potential flow analysis while solving real fluid flow problems.
- The **Prandtl boundary layer equations** are thus a simplification of the Navier-Stokes equations.

Boundary Layer Coordinates

- The boundary layer equations derived are in Cartesian coordinates.
- The Velocity components u and v represent x and y direction velocities respectively.
- For objects with small curvature, these equations can be used with -
    - x coordinate : streamwise direction
    - y coordinate : normal component
- They are called **Boundary Layer Coordinates.**

Application of Boundary Layer Theory

- The Boundary-Layer Theory is not valid beyond the point of separation.
- At the point of separation, boundary layer thickness becomes quite large for the thin layer approximation to be valid.
- It is important to note that boundary layer theory can be used to locate the point of seperation itself.
- In applying the boundary layer theory although U is the free-stream velocity at the outer edge of the boundary layer, it is interpreted as the fluid velocity at the wall calculated from inviscid flow considerations (known as **Potential Wall Velocity**)
- Mathematically, application of the boundary - layer theory converts the character of governing Navier-Stroke equations from elliptic to parabolic
- This allows the marching in flow direction, as the solution at any location is independent of the conditions farther downstream.

Blasius Flow Over A Flat Plate

- The classical problem considered by H. Blasius was
    1. Two-dimensional, steady, incompressible flow over a flat plate at zero angle of incidence with respect to the uniform stream of velocity U_∞ .

2. The fluid extends to infinity in all directions from the plate.

The physical problem is already illustrated in Fig. 1

- Blasius wanted to determine
  (a) the velocity field solely within the boundary layer,
  (b) the boundary layer thickness (δ),
  (c) the shear stress distribution on the plate, and
  (d) the drag force on the plate.
- The Prandtl boundary layer equations in the case under consideration are

$$u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \quad (15)$$

$$\nu=\mu/\rho$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

The boundary conditions are

$$at\ y=0,\ u=v=0 \quad (16)$$

$$at\quad y=\infty,\quad u=U_\infty$$

- Note that the substitution of the term $-\frac{1}{\rho}\frac{dp}{dx}$ in the original boundary layer momentum equation in terms of the free stream velocity produces $U_\infty\frac{dU_\infty}{dx}$ which is equal to zero.
- Hence the governing Eq. (15) does not contain any pressure-gradient term.
- However, the characteristic parameters of this problem are U_∞, ν, x, y that is, $u=u(U_\infty, \nu, x, y)$
- This relation has five variables U_∞, ν, x, y **.**
- It involves two dimensions, length and time.
- Thus it can be reduced to a dimensionless relation in terms of (5-2) =3 quantities (**Buckingham Pi Theorem**)
- Thus a similarity variables can be used to find the solution
- Such flow fields are called **self-similar flow field .**

Law of Similarity for Boundary Layer Flows

- It states that the *u* component of velocity with two velocity profiles of *u(x,y)* at different *x* locations differ only by scale factors in *u* and *y* .

- Therefore, the velocity profiles *u(x,y)* at all values of *x* can be made congruent if they are plotted in coordinates which have been made dimensionless with reference to the scale factors.
- The local free stream velocity *U(x)* at section *x* is an obvious scale factor for *u*, because the dimensionless *u(x)* varies between zero and unity with *y* at all sections.
- The scale factor for *y* , denoted by *g(x)* , is proportional to the local boundary layer thickness so that *y* itself varies between zero and unity.
- Velocity at two arbitrary *x* locations, namely x_1 and x_2 should satisfy the equation

$$\frac{u[x_1,\{y/g(x_1)\}]}{U(x_1)} = \frac{u[x_2,\{y/g(x_2)\}]}{U(x_2)} \tag{17}$$

- Now, for Blasius flow, it is possible to identify g(x) with the boundary layers thickness δ we know

$$\varepsilon = \frac{\delta}{L} \sim \frac{1}{\sqrt{\mathrm{Re}_L}}$$

Thus in terms of x we get

$$\frac{\delta}{x} \sim \frac{1}{\sqrt{\frac{U_\infty x}{\nu}}}$$

$$\delta \sim \sqrt{\frac{\nu x}{U_\infty}}$$

i.e.,

$$\frac{u}{U_\infty} = F\left(\frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}\right) = F(\eta) \tag{18}$$

where $\eta \sim \frac{y}{\delta}$ *and* $\delta \sim \sqrt{\frac{\nu x}{U_\infty}}$

or more precisely,

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}} \tag{19}$$

$$y = \eta\sqrt{\frac{\nu x}{U_\infty}}$$

$$dy = \sqrt{\frac{\nu x}{U_\infty}}d\eta$$

The stream function can now be obtained in terms of the velocity components as

$$\psi = \int u dy = \int U_\infty F(\eta)\sqrt{\frac{\nu x}{U_\infty}}d\eta = \sqrt{U_\infty \nu x}\int F(\eta)d\eta$$

Or

$$\psi = \sqrt{U_\infty \nu x} f(\eta) + \mathrm{D} \tag{20}$$

where D is a constant. Also $\int F(\eta)d\eta = f(\eta)$ and the constant of integration is zero if the stream function at the solid surface is set equal to zero.

Now, the velocity components and their derivatives are:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}.\frac{\partial \eta}{\partial y} = U_\infty f'(\eta)$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{U_\infty \nu}\left[\frac{1}{2}.\frac{1}{\sqrt{x}}f(\eta) + \sqrt{x}f'(\eta)\left\{-\frac{1}{2}\frac{y}{\sqrt{\nu x / U_\infty}}\frac{1}{x}\right\}\right]$$

or

$$v = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left[\eta f'(\eta) - f(\eta)\right] \tag{21}$$

$$\frac{\partial u}{\partial x} = U_\infty f''(\eta)\frac{\partial \eta}{\partial x} = U_\infty f''(\eta)\left[-\frac{1}{2}.\frac{y}{\sqrt{\nu x / U_\infty}}\frac{1}{x}\right]$$

$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2}.\frac{\eta}{x}.f''(\eta) \tag{22}$$

$$\frac{\partial u}{\partial y} = U_\infty f''(\eta)\frac{\partial \eta}{\partial y} = U_\infty f''(\eta)\left[\frac{1}{\sqrt{\nu x / U_\infty}}\right]$$

$$\frac{\partial u}{\partial y} = U_\infty \sqrt{\frac{U_\infty}{\nu x}} f''(\eta) \qquad (23)$$

$$\frac{\partial^2 u}{\partial y^2} = U_\infty \sqrt{\frac{U_\infty}{\nu x}} f''(\eta)\left\{\frac{1}{\sqrt{\nu x / U_\infty}}\right\}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{\nu x} f'''(\eta) \qquad (24)$$

- Substituting (28.2) into (28.15), we have

$$-\frac{U_\infty^2}{2}\frac{\eta}{x}.f'(\eta)f''(\eta) + \frac{U_\infty^2}{2x}\left[\eta f'(\eta) - f(\eta)\right]f''(\eta) = \frac{U_\infty^2}{x} f'''(\eta)$$

$$-\frac{1}{2}\frac{U_\infty^2}{x} f(\eta)f''(\eta) = \frac{U_\infty^2}{x} f'''(\eta)$$

or,

$$2f'''(\eta) + f(\eta)f''(\eta) = 0 \qquad (25)$$

where

$$f(\eta) = \int F(\eta)d\eta + C$$

$$= \int \frac{u}{U_\infty} d\eta + C$$

and

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}$$

This is known as Blasius Equation .

- The boundary conditions as in Eg. (28.16), in combination with Eg. (28.21a) and (28.21b) become

at $y = 0, u = 0$, therefore $\eta = 0 : f(\eta) = 0, f'(\eta) = 0$

at $y = \infty, u = U_\infty$ therefore $\eta = \infty : F(\eta) = f'(\eta) = 1$	(26)

Equation (22) is a **third order nonlinear differential equation .**

- Blasius obtained the solution of this equation in the form of series expansion through analytical techniques
- We shall not discuss this technique. However, we shall discuss a numerical technique to solve the aforesaid equation which can be understood rather easily.
- Note that the **equation for** f **does not contain** x .
- **Boundary conditions at** $x = 0$ **and** $y = \infty$ **merge into the condition** $\eta \to \infty, u / U_\infty = f' = 1$ **. This is the key feature of similarity solution.**
- We can rewrite Eq. (28.22) as three first order differential equations in the following way

$f' = G$	(27)
$G' = H$	(28)
$H' = -\frac{1}{2} f H$	(29)

- Let us next consider the boundary conditions.

1. The condition $f(0) = 0$ remains valid.
2. The condition $f'(0) = 0$ means that $G(0) = 0$ **.**
3. The condition $f'(\infty) = 1$ gives us $G(\infty) = 1$.

Note that the equations for *f* and *G* have initial values. However, the value for H(0) is not known. Hence, we do not have a usual initial-value problem.

Shooting Technique

We handle this problem as an initial-value problem by choosing values of $H(0)$ and solving by numerical methods $f(\eta), G(\eta)$, and $H(\eta)$.

In general, the condition $G(\infty) = 1$ will not be satisfied for the function G arising from the numerical solution.

We then choose other initial values of H so that eventually we find an $H(0)$ which results in $G(\infty) = 1$.

This method is called the shooting technique **.**

- In Eq. (28.24), the primes refer to differentiation wrt. the similarity variable η . The integration steps following Runge-Kutta method are given below.

$f_{n+1} = f_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$	(30)
$G_{n+1} = G_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$	(31)
$H_{n+1} = H_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$	(32)

- One moves from η_n to $\eta_{n+1} = \eta_n + h$. A fourth order accuracy is preserved if *h* is constant along the integration path, that is, $\eta_{n+1} - \eta_n = h$ for all values of *n* . The values of *k, l* and *m* are as follows.
- For generality let the system of governing equations be

$$f' = F_1(f, G, H, \eta), G' = F_2(f, G, H, \eta) \ \& \ H' = F_3(f, G, H, \eta)$$

$$k_1 = hF_1(f_n, G_n, H_n, \eta_n)$$

$$l_1 = hF_2(f_n, G_n, H_n, \eta_n)$$

$$m_1 = hF_3(f_n, G_n, H_n, \eta_n)$$

$$k_2 = hF_1\left\{\left((f_n + \frac{1}{2}k_1), (G_n + \frac{1}{2}l_1), (H_n + \frac{1}{2}m_1), (\eta_n + \frac{h}{2})\right)\right\}$$

$$l_2 = hF_2\left\{\left((f_n + \frac{1}{2}k_1), (G_n + \frac{1}{2}l_1), (H_n + \frac{1}{2}m_1), (\eta_n + \frac{h}{2})\right)\right\}$$

$$m_2 = hF_3\left\{\left((f_n + \frac{1}{2}k_1), (G_n + \frac{1}{2}l_1), (H_n + \frac{1}{2}m_1), (\eta_n + \frac{h}{2})\right)\right\}$$

In a similar way K_3, l_3, m_3 and k_4, l_4, m_4 mare calculated following standard formulae for the Runge-Kutta integration. For example, K_3 is given by

$$k_3 = hF_1\left\{\left((f_n + \frac{1}{2}k_2), (G_n + \frac{1}{2}l_2), (H_n + \frac{1}{2}m_2), (\eta_n + \frac{h}{2})\right)\right\}$$

The functions F_1, $F_2$and F_3 are G, H , - *f H / 2* respectively. Then at a distance $\Delta\eta$ from the wall, we have

$$f(\Delta\eta) = f(0) + G(0)\Delta\eta \quad (33)$$

$$G(\Delta\eta) = G(0) + H(0)\Delta\eta \quad (34)$$

$$H(\Delta\eta) = H(0) + H'(0)\Delta\eta \quad (35)$$

$$H'(\Delta\eta) = -\frac{1}{2}f(\Delta\eta)H(\Delta\eta) \quad (36)$$

- As it has been mentioned earlier $f''(0) = H(0) = \lambda$ is unknown. It must be determined such that the condition $f'(\infty) = G(\infty) = 1$ is satisfied.

The condition at infinity is usually approximated at a finite value of η (around $\eta = 10$). The process of obtaining λ accurately involves iteration and may be calculated using the procedure described below.

- For this purpose, consider Fig. 28.2(a) where the solutions of G versus η for two different values of $H(0)$ are plotted. The values of $G(\infty)$ are estimated from the G curves and are plotted in Fig. 28.2(b).
- The value of $H(0)$ now can be calculated by finding the value $\widetilde{H}(0)$ at which the line 1-2 crosses the line $G(\infty) = 1$ By using similar triangles, it can be said that $\frac{\widetilde{H}(0) - H(0)_1}{1 - G(\infty)_1} = \frac{H(0)_2 - H(0)_1}{G(\infty)_2 - G(\infty)_1}$. By solving this, we get $\widetilde{H}(0)$.
- Next we repeat the same calculation as above by using $\widetilde{H}(0)$ and the better of the two initial values of $H(0)$. Thus we get another improved value $\widetilde{\widetilde{H}}(0)$. This process may continue, that is, we use $\widetilde{\widetilde{H}}(0)$ and $\widetilde{H}(0)$ as a pair of values to find more improved

values for $H(0)$, and so forth. The better guess for H (0) can also be obtained by using the Newton Raphson Method. It should be always kept in mind that for each value of $H(0)$, the curve $G(\eta)$ versus η is to be examined to get the proper value of $G(\infty)$.

- The functions $f(\eta), f'(\eta) = G$ and $f''(\eta) = H$ are plotted in Fig. 28.3.The velocity components, *u* and *v* inside the boundary layer can be computed from Eqs (28.21a) and (28.21b) respectively.
- A sample computer program in FORTRAN follows in order to explain the solution procedure in greater detail. The program uses Runge Kutta integration together with the Newton Raphson method

Download the program

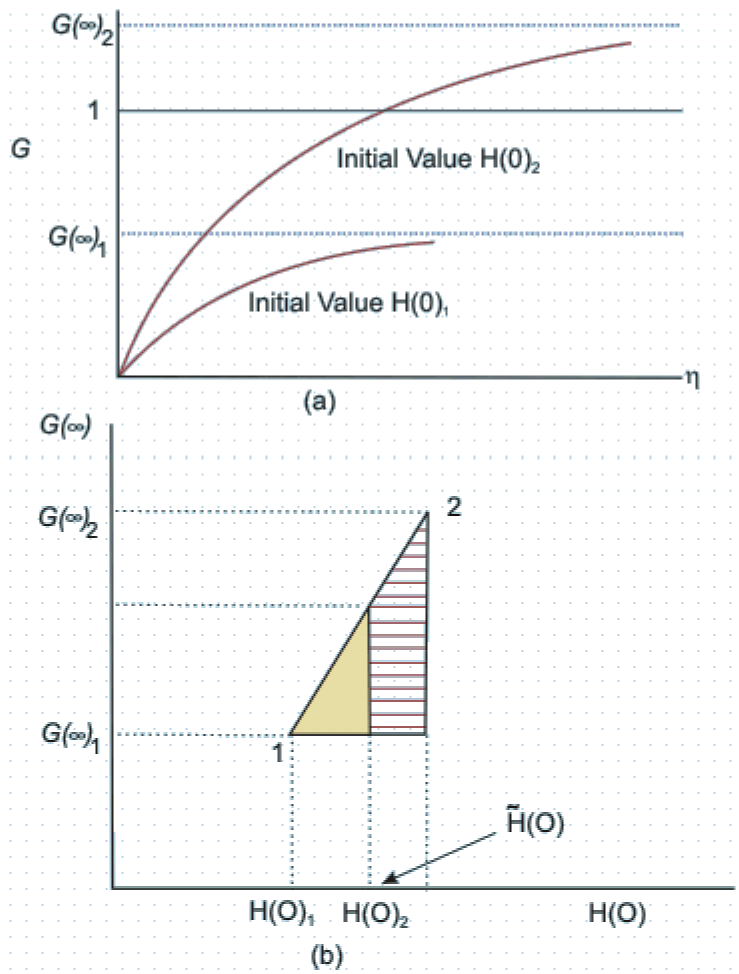

Fig 2 Correcting the initial guess for H(O)

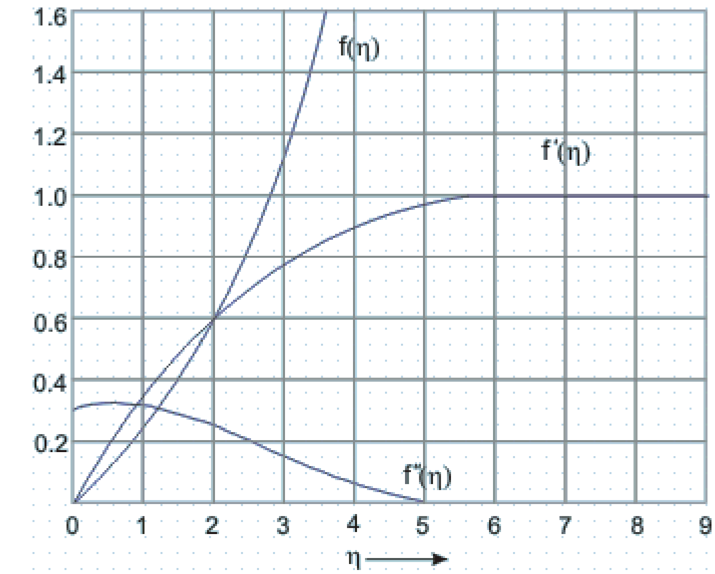

Fig 3 f, G and H distribution in the boundary layer

- Measurements to test the accuracy of theoretical results were carried out by many scientists. In his experiments, J. Nikuradse, found excellent agreement with the theoretical results with respect to velocity distribution (u/U_∞) within the boundary layer of a stream of air on a flat plate.
- In the next slide we'll see some values of the velocity profile shape $f'(\eta) = u/U_\infty = G$ and $f''(\eta) = H$ in tabular format.

Lecture Note 2

N – S Equations

Navier-Strokes Equation

- Generalized equations of motion of a real flow named after the inventors CLMH Navier and GG Stokes are derived from the **Newton's second law**
- **Newton's second law** states that the **product of mass and acceleration is equal to sum of the external forces acting on a body.**
- External forces are of two kinds-
    - one acts throughout the mass of the body ----- **body force** (gravitational force, electromagnetic force)
    - another acts on the boundary----- **surface force** (pressure and frictional force).

Objective - We shall consider a differential fluid element in the flow field (Fig.1). Evaluate the surface forces acting on the boundary of the rectangular parallelepiped shown below.

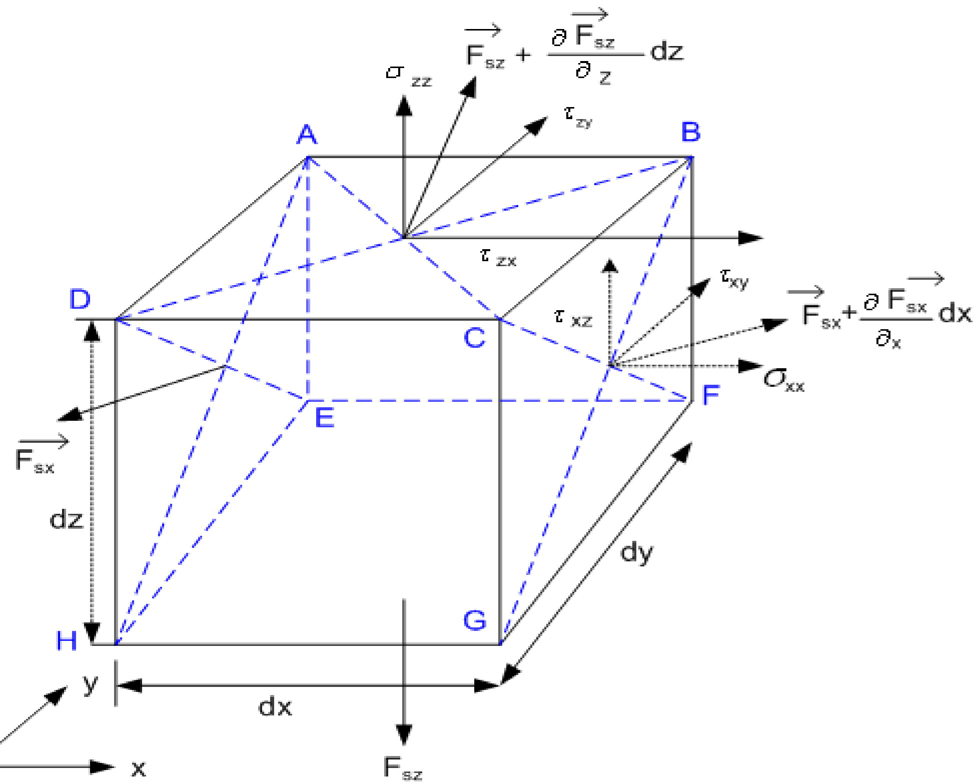

Fig.1 Definition of the components of stress and their locations in a differential fluid element

- Let the body force per unit mass be

$$\vec{f}_b = \hat{i}\ f_x + \hat{j}\ f_y + \hat{k}\ f_z \tag{1}$$

and surface force per unit volume be

$$\vec{F} = \hat{i}\ F_x + \hat{j}\ F_y + \hat{k}\ F_z \tag{2}$$

- Consider surface force on the surface AEHD, per unit area,

$$\vec{F}_{sx} = \hat{i}\ \sigma_{xx} + \hat{j}\ \tau_{xy} + \hat{k}\ \tau_{xz}$$

[Here second subscript x denotes that the surface force is evaluated for the surface whose outward normal is the x axis]

- Surface force on the surface *BFGC* per unit area is

$$\vec{F}_{sx} + \frac{\partial \vec{F}_{sx}}{\partial x} dx$$

- Net force on the body due to imbalance of surface forces on the above two surfaces is

$$\frac{\partial \vec{F}_{sx}}{\partial x} dx\ dy\ dz \tag{3}$$

(since area of faces AEHD and BFGC is dydz)

- Total force on the body due to net surface forces on all six surfaces is

$$\left(\frac{\partial \vec{F}_{sx}}{\partial x} + \frac{\partial \vec{F}_{sy}}{\partial y} + \frac{\partial \vec{F}_{sz}}{\partial z} \right) dx\ dy\ dz \tag{4}$$

- And hence, the resultant surface force *dF*, per unit volume, is

$$d\vec{F} = \frac{\partial \vec{F}_{sx}}{\partial x} + \frac{\partial \vec{F}_{sy}}{\partial y} + \frac{\partial \vec{F}_{sz}}{\partial z} \tag{5}$$

(since Volume= d*x* d*y* dz)

- The quantities $\vec{F}_{sx}$, $\vec{F}_{sy}$ and $\vec{F}_{sz}$ are vectors which can be resolved into normal stresses denoted by σ and shearing stresses denoted by τ as

$$\vec{F}_{sx} = \hat{i}\ \sigma_{xx} + \hat{j}\ \tau_{xy} + \hat{k}\ \tau_{xz} \tag{6}$$

$$\vec{F}_{sy} = \hat{i}\,\tau_{yx} + \hat{j}\,\sigma_{yy} + \hat{k}\,\tau_{yz}$$
$$\vec{F}_{sz} = \hat{i}\,\tau_{zx} + \hat{j}\,\tau_{zy} + \hat{k}\,\sigma_{zz}$$

The **stress system has nine scalar quantities. These nine quantities form a stress tensor.**

Nine Scalar Quantities of Stress System - Stress Tensor

The set of nine components of stress tensor can be described as

$$\pi = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \qquad (7)$$

- The stress tensor is ***symmetric***,
- This means that two shearing stresses with subscripts which differ only in their sequence are equal. For example $\tau_{xz} = \tau_{zx}$
- Considering the equation of motion for instantaneous rotation of the fluid element (Fig. 24.1) about y axis, we can write

$$\dot{\omega}_y\, dI_y = (\tau_{xz}\, dy\, dz)dx - (\tau_{zx}\, dx\, dy)dz$$
$$= (\tau_{xz} - \tau_{zx})\, d\forall$$

where $d\forall$ =dxdydz is the volume of the element, $\dot{\omega}_y$ is the angular acceleration

dI_y is the moment of inertia of the element about y-axis

- Since dI_y is proportional to fifth power of the linear dimensions and $d\forall$ is proportional to the third power of the linear dimensions, the left hand side of the above equation and the second term on the right hand side vanishes faster than the first term on the right hand side on contracting the element to a point.
- Hence, the result is

$$\tau_{xz} = \tau_{zx}$$

From the similar considerations about other two remaining axes, we can write

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

which has already been observed in Eqs (24.2a), (24.2b) and (24.2c) earlier.

- Invoking these conditions into Eq. (24.12), the stress tensor becomes

$$\pi = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \quad (8)$$

- Combining Eqs (24.10), (24.11) and (24.13), the resultant surface force per unit volume becomes

$$d\overline{F} = \hat{i}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) + \hat{j}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) + \hat{k}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) \quad (9)$$

- As per the velocity field,

$$\frac{D\vec{V}}{Dt} = \hat{i}\frac{Du}{Dt} + \hat{j}\frac{D\upsilon}{Dt} + \hat{k}\frac{Dw}{Dt} \quad (10)$$

By Newton's law of motion applied to the differential element, we can write

$$\rho(dx\,dy\,dz)\frac{D\vec{V}}{Dt} = \left(d\vec{F}\right)(dx\,dy\,dz) + \rho\vec{f}_b(dx\,dy\,dz)$$

or,
$$\rho\frac{D\vec{V}}{Dt} = d\vec{F} + \rho\,\vec{f}_b$$

Substituting Eqs (24.15), (24.14) and (24.6) into the above expression, we obtain

$$\rho\frac{Du}{Dt} = \rho f_x + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) \quad (11)$$

$$\rho\frac{D\upsilon}{Dt} = \rho f_y + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) \tag{12}$$

$$\rho\frac{Dw}{Dt} = \rho f_z + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) \tag{13}$$

Since

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\nabla\cdot\vec{V}\right)$$

$$\frac{\partial \sigma_{xx}}{\partial x} = -\,{\partial p}/{\partial x} + {\partial}/{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right]$$

Similarly others follow.

- So we can express $\frac{Du}{Dt}$, $\frac{D\upsilon}{Dt}$ and $\frac{Dw}{Dt}$ in terms of field derivatives,

$$\rho\frac{Du}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \tag{14}$$

$$\rho\frac{D\upsilon}{Dt} = \rho f_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial \upsilon}{\partial y} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial \upsilon}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right)\right] \tag{15}$$

$$\rho\frac{Dw}{Dt} = \rho f_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial \upsilon}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \tag{16}$$

- These differential equations are known as Navier-Stokes equations.
- At this juncture, discuss the equation of continuity as well, which has a **general (conservative) form**

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho \upsilon)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{17}$$

- In case of incompressible flow $\rho =$ constant. Therefore, equation of continuity for incompressible flow becomes

$$\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (18)$$

- Invoking Eq. (24.19) into Eqs (24.17a), (24.17b) and (24.17c), we get

$$\rho\frac{Du}{Dt}=\rho f_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}\right)\right]+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\mu\left(\frac{\partial}{\partial y}\frac{\partial v}{\partial x}\right)+\mu\left(\frac{\partial}{\partial z}\frac{\partial w}{\partial x}\right)$$

$$=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\cancelto{0}{\frac{\partial v}{\partial y}}+\frac{\partial w}{\partial z}\right)$$

Similarly others follow

Thus,

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad (19)$$

$$\rho\left(\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}+w\frac{\partial \upsilon}{\partial z}\right)=\rho f_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 \upsilon}{\partial x^2}+\frac{\partial^2 \upsilon}{\partial y^2}+\frac{\partial^2 \upsilon}{\partial z^2}\right) \qquad (20)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+\upsilon\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho f_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \qquad (21)$$

Vector Notation & derivation in Cylindrical Coordinates - Navier-Stokes equation

- Using, vector notation to write Navier-Stokes and continuity equations for incompressible flow we have

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{f}_b-\nabla p+\mu\nabla^2\vec{V} \qquad (22)$$

And

$$\nabla\cdot\vec{V}=0 \qquad (23)$$

- we have **four unknown quantities, u, v, w and p** ,
- we also have **four equations, - equations of motion in three directions and the continuity equation.**
- In principle, these equations are solvable but to date generalized solution is not available due to the complex nature of the set of these equations.

- The highest order terms, which come from the viscous forces, are linear and of second order
- The first order convective terms are non-linear and hence, the set is termed as **quasi-linear.**
- Navier-Stokes equations in cylindrical coordinate (Fig. 24.2) are useful in solving many problems. If , and denote the velocity components along the radial, cross-radial and axial directions respectively, then for the case of incompressible flow, Eqs (24.21) and (24.22) lead to the following system of equations:

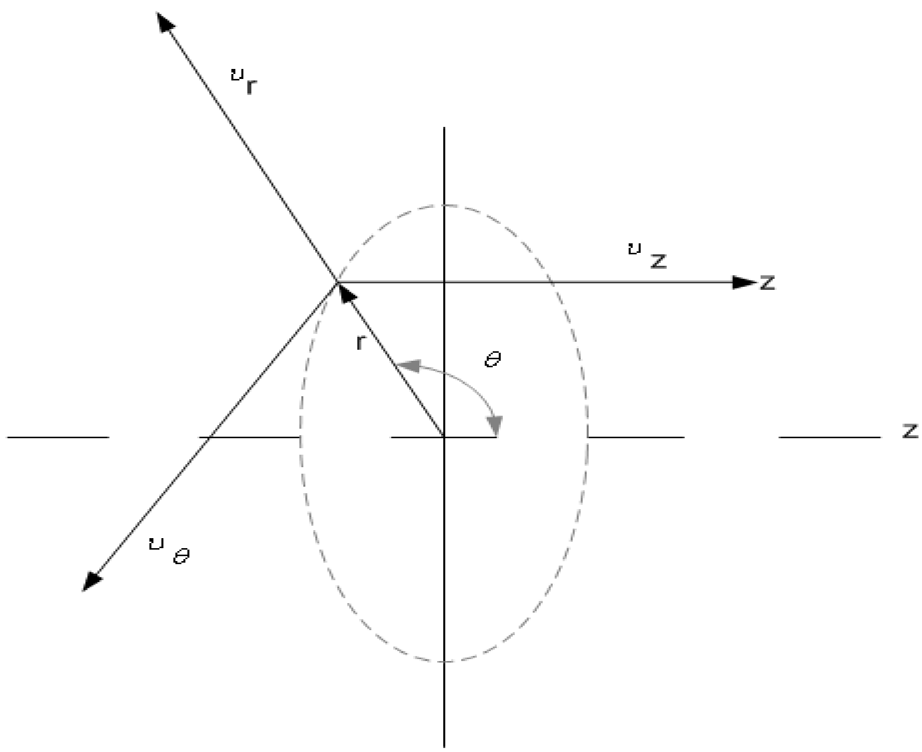

FIG 2 Cylindrical polar coordinate and the velocity components

$$\rho\left(\frac{\partial \upsilon_r}{\partial t}+\upsilon_r\frac{\partial \upsilon_r}{\partial r}+\frac{\upsilon_\theta}{r}\cdot\frac{\partial \upsilon_r}{\partial \theta}-\frac{\upsilon_\theta^2}{r}+\upsilon_z\frac{\partial \upsilon_r}{\partial z}\right)=\rho f_r-\frac{\partial p}{\partial r}$$
$$+\mu\left(\frac{\partial^2 \upsilon_r}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial \upsilon_r}{\partial r}-\frac{\upsilon_r}{r^2}+\frac{1}{r^2}\cdot\frac{\partial^2 \upsilon_r}{\partial \theta^2}-\frac{2}{r^2}\cdot\frac{\partial \upsilon_\theta}{\partial \theta}+\frac{\partial^2 \upsilon_r}{\partial z^2}\right) \qquad (24)$$

$$\rho\left(\frac{\partial \upsilon_\theta}{\partial t}+\upsilon_r\frac{\partial \upsilon_\theta}{\partial r}+\frac{\upsilon_\theta}{r}\cdot\frac{\partial \upsilon_\theta}{\partial \theta}+\upsilon_z\frac{\partial \upsilon_\theta}{\partial z}+\frac{\upsilon_r\upsilon_\theta}{r}\right)=\rho f_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+\mu\left(\frac{\partial^2 \upsilon_\theta}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial \upsilon_\theta}{\partial r}-\frac{\upsilon_\theta}{r^2}+\frac{1}{r^2}\cdot\frac{\partial^2 \upsilon_\theta}{\partial \theta^2}+\frac{2}{r^2}\cdot\frac{\partial \upsilon_r}{\partial \theta}+\frac{\partial^2 \upsilon_\theta}{\partial z^2}\right) \quad (25)$$

$$\rho\left(\frac{\partial \upsilon_z}{\partial t}+\upsilon_r\frac{\partial \upsilon_z}{\partial r}+\frac{\upsilon_\theta}{r}\cdot\frac{\partial \upsilon_z}{\partial \theta}+\upsilon_z\frac{\partial \upsilon_z}{\partial z}\right)=\rho f_z-\frac{\partial p}{\partial z}$$
$$+\mu\left(\frac{\partial^2 \upsilon_z}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial \upsilon_z}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 \upsilon_z}{\partial \theta^2}+\frac{\partial^2 \upsilon_z}{\partial z^2}\right) \quad (26)$$

$$\frac{\partial \upsilon_r}{\partial r}+\frac{\upsilon_r}{r}+\frac{1}{r}\frac{\partial \upsilon_\theta}{\partial \theta}+\frac{\partial \upsilon_z}{\partial z}=0 \quad (27)$$

A general way of deriving the Navier-Stokes equations from the basic laws of physics.

- Consider a general flow field as represented in Fig. 25.1.
- Imagine a closed control volume, $\forall_0$ within the flow field. The control volume is fixed in space and the fluid is moving through it. The control volume occupies reasonably large finite region of the flow field.
- A control surface , A_0 is defined as the surface which bounds the volume $\forall_0$.
- According to Reynolds transport theorem, "The rate of change of momentum for a system equals the sum of the rate of change of momentum inside the control volume and the rate of efflux of momentum across the control surface".
- The rate of change of momentum for a system (in our case, the control volume boundary and the system boundary are same) is equal to the net external force acting on it.

Now, we shall transform these statements into equation by accounting for each term,

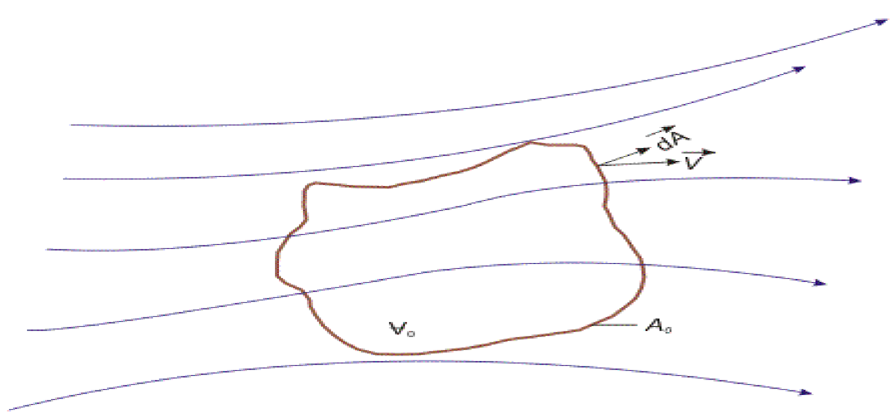

FIG 25.1 Finite control volume fixed in space with the fluid moving through it

- Rate of change of momentum inside the control volume

$$= \frac{\partial}{\partial t}\int\!\!\int\!\!\int_{\forall_0} \rho \vec{V} d\forall$$

$$= \int\!\!\int\!\!\int_{\forall_0} \frac{\partial}{\partial t}\left(\rho \vec{V}\right) d\forall \qquad (28)$$

(**since t is independent of space variable**)

- **Rate of efflux of momentum through control surface**

$$\int\!\!\int_{A_0} \rho \vec{V}\left(\vec{V}.d\vec{A}\right) = \int\!\!\int_{A_0} \rho \vec{V}\vec{V}.\vec{n} dA$$

$$= \int\!\!\int\!\!\int_{\forall_0} \left(\vec{V}\left(\nabla.\rho \vec{V}\right) + \rho \vec{V}.\nabla \vec{V}\right) d\forall \qquad (29)$$

- **Surface force acting on the control volume**

``

$$= \int\!\!\int\!\!\int_{\forall_0} \left(\nabla.\sigma\right) d\forall \qquad (30)$$

- **Body force acting on the control volume**

$$\int\!\!\int\!\!\int_{\forall_0} \rho \vec{f}_b d\forall \qquad (31)$$

$\vec{f}_b$ in Eq. (25.4) is the body force per unit mass.

- **Finally, we get,**

$$\iiint_{\forall_0} \left(\frac{\partial}{\partial t}(\rho\vec{V}) + \{\vec{V}(\nabla.\rho\vec{V}) + \rho\vec{V}.\nabla\vec{V}\} \right) d\forall$$

$$= \iiint_{\forall_0} \left(\nabla.\sigma + \rho\vec{f}_b\right) d\forall$$

or

or, $$\rho\frac{\partial\vec{V}}{\partial t} + \vec{V}\frac{\partial\rho}{\partial t} + \rho\vec{V}.\nabla\vec{V} + \vec{V}(\nabla.\rho\vec{V}) = \nabla.\sigma + \rho\,\vec{f}_b$$

or $$\rho\left(\frac{\partial\vec{V}}{\partial t} + \vec{V}.\nabla\vec{V}\right) + \vec{V}\left(\frac{\partial\rho}{\partial t} + \nabla.\rho\vec{V}\right) = \nabla.\sigma + \rho\,\vec{f}_b \qquad (32)$$

We know that $\frac{\partial\rho}{\partial t} + \nabla.\rho\vec{V} = 0$ is the general form of **mass conservation equation** (popularly known as the **continuity equation**), valid for both **compressible** and **incompressible** flows.

- Invoking this relationship in Eq. (25.5), we obtain

$$\rho\left(\frac{\partial\vec{V}}{\partial t} + \vec{V}.\nabla\vec{V}\right) = \nabla.\sigma + \rho\,\vec{f}_b$$

or $$\rho\frac{D\vec{V}}{Dt} = \nabla.\sigma + \rho\,\vec{f}_b \qquad (33)$$

- Equation (25.6) is referred to as **Cauchy's equation of motion** . In this equation, σ is the stress tensor,
- After having substituted σ we get

$$\nabla.\sigma = -\nabla p + (\mu' + \mu)\nabla(\nabla.\vec{V}) + \mu\nabla^2\vec{V} \qquad (34)$$

From **Stokes's hypothesis** we get, $\mu' + \frac{2}{3}\mu = 0 \qquad (35)$

Invoking above two relationships into Eq.(25.6) we get

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \mu \nabla^2 \vec{V} + \frac{1}{3}\mu \nabla\left(\nabla . \vec{V}\right) + \rho \vec{f}_b \qquad (36)$$

This is the most general form of Navier-Stokes equation.

Exact Solutions Of Navier-Stokes Equations

Consider a class of flow termed as parallel flow in which **only one velocity term is nontrivial** and all the fluid particles move in one direction only.

- We **choose** x **to be the direction along which all fluid particles travel** , i.e. $u \neq 0,\ v = w = 0$. Invoking this in continuity equation, we get

$$\frac{\partial u}{\partial x} + \cancelto{0}{\frac{\partial v}{\partial y}} + \cancelto{0}{\frac{\partial w}{\partial z}} = 0$$

$$\frac{\partial u}{\partial z} = 0 \quad \text{which means} \quad u = u\left(y, z, t\right)$$

- Now. Navier-Stokes equations for incompressible flow become

$$\frac{\partial u}{\partial t} + u\cancelto{0}{\frac{\partial u}{\partial x}} + v\cancelto{0}{\frac{\partial u}{\partial y}} + w\cancelto{0}{\frac{\partial u}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\cancelto{0}{\frac{\partial v}{\partial t}} + u\cancelto{0}{\frac{\partial v}{\partial x}} + v\cancelto{0}{\frac{\partial v}{\partial y}} + w\cancelto{0}{\frac{\partial v}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\cancelto{0}{\frac{\partial^2 v}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v}{\partial z^2}}\right]$$

$$\cancelto{0}{\frac{\partial w}{\partial t}} + u\cancelto{0}{\frac{\partial w}{\partial x}} + v\cancelto{0}{\frac{\partial w}{\partial y}} + w\cancelto{0}{\frac{\partial w}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\cancelto{0}{\frac{\partial^2 w}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 w}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 w}{\partial z^2}}\right]$$

So, we obtain

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial z} = 0 \quad \text{which means } p = p\left(x\right) \textit{ alone}$$

and $\frac{\partial u}{\partial t} = -\frac{1}{\rho}.\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$	(37)

Parallel Flow in a Straight Channel

Consider **steady flow between two infinitely broad parallel plates** as shown in Fig. 25.2.

Flow is independent of any variation in z direction, hence, z dependence is gotten rid of and Eq. (25.11) becomes

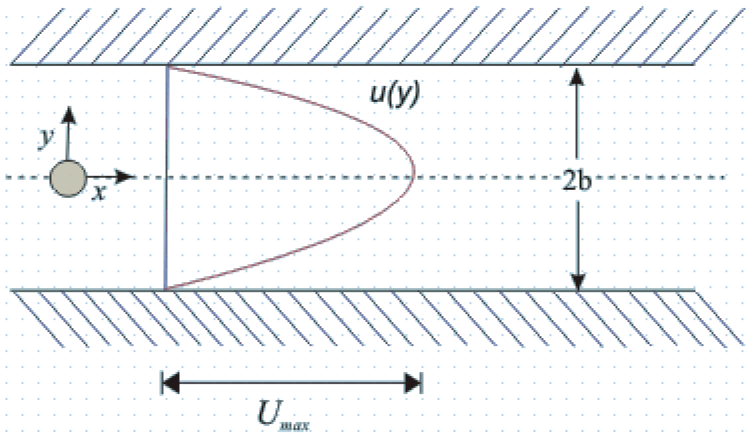

FIG 25.2 Parallel flow in a straight channel

$\frac{dp}{dx} = \mu\frac{\partial^2 u}{\partial y^2}$	(38)

The boundary conditions are at y = b, u = 0; and y = -b, u = O.

- From Eq. (25.12), we can write

$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1$	
or $u = \frac{1}{2\mu}.\frac{dp}{dx}y^2 + C_1 y + C_2$	

- Applying the boundary conditions, the constants are evaluated as

$C_1 = 0$ and $C_2 = -\frac{1}{\mu}.\frac{dp}{dx}.\frac{b^2}{2}$	

So, the solution is

$$u = -\frac{1}{2\mu}.\frac{dp}{dx}\left(b^2 - y^2\right) \qquad (39)$$

which implies that the velocity profile is parabolic.

Average Velocity and Maximum Velocity

- To establish the relationship between the maximum velocity and average velocity in the channel, we analyze as follows

  At y = 0, $u = U_{max}$; this yields

$$U_{max} = -\frac{b^2}{2\mu}.\frac{dp}{dx} \qquad (40)$$

On the other hand, the average velocity,

$$U_{av} = \frac{Q}{2b} = \frac{\text{flow rate}}{\text{flow area}} = \frac{1}{2b}\int_{-b}^{b} u\,dy$$

or $$U_{av} = \frac{2}{2b}\int_{0}^{b} -\frac{1}{2\mu}.\frac{dp}{dx}\left(b^2 - y^2\right)dy$$

$$= \frac{1}{2\mu}\frac{dp}{dx}.\frac{1}{b}\left\{\left[b^2 y\right]_0^b - \left(\frac{y^3}{3}\right)_0^b\right\}$$

Finally, $$U_{av} = -\frac{1}{2\mu}\frac{dp}{dx}.\frac{2}{3}b^2 \qquad (41)$$

So, $\dfrac{U_{av}}{U_{max}} = \dfrac{2}{3}$ or $U_{max} = \dfrac{3}{2}U_{av}$ (42)

- The shearing stress at the wall for the parallel flow in a channel can be determined from the velocity gradient as

$$\tau_{yx}\big|_b = \mu\left(\frac{\partial u}{\partial y}\right)_b = b\frac{dp}{dx} = -2\mu\frac{U_{max}}{b}$$

Since the upper plate is a "minus y surface", a negative stress acts in the positive x direction, i.e. to the right.

- The local friction coefficient, C_f is defined by

$$C_f = \frac{\left|\left(\tau_{yx}\right)_b\right|}{\frac{1}{2}\rho U_{av}^2} = \frac{3\mu U_{av}/b}{\frac{1}{2}\rho U_{av}^2}$$

$$C_f = \frac{12}{\frac{\rho U_{av}(2b)}{\mu}} = \frac{12}{Re} \tag{43}$$

where $Re = U_{av}(2b)/\nu$ is the Reynolds number of flow based on average velocity and the channel height (2b).

- Experiments show that Eq. (25.14d) is valid in the laminar regime of the channel flow.
- The maximum Reynolds number value corresponding to fully developed laminar flow, for which a stable motion will persist, is 2300.
- In a reasonably careful experiment, laminar flow can be observed up to even Re = 10,000.
- But the value below which the flow will always remain laminar, i.e. **the critical value of Re is 2300.**

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